STELLATION OF POLYHEDRA, AND COMPUTER IMPLEMENTATION

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Abstract. The construction and display of graphic representations of stellated polyhedra requires a painstaking and close analysis of the geometrical objects generated by extending the facial planes of the polyhedron. This analysis also facilitates efficient solution of the problem of counting the stellations of a polyhedron.

1. Introduction

Stellation is a process of extending the faces of a polyhedron, called the **base**, to form a new, usually more complex, polyhedron. The results include many interesting and beautiful figures. For example, three of the four non-convex regular polyhedra are stellations of the dodecahedron, and the fourth is a stellation of the icosahedron. Of the five regular compound polyhedra, one is a stellation of the octahedron, three are stellations of the icosahedron, and the fifth is a stellation of the rhombic triacontahedron.

1.1. Choice of a Base.

Stellation is most rewarding when it starts from a base polyhedron with a high degree of symmetry. A thorough study of such polyhedra may be found in [5]; one may also consult [1], Chap. V.

Any discussion of symmetrical polyhedra has to begin with the **regular** or **Platonic** polyhedra. A Platonic polyhedron is convex; its faces are regular polygons; any two faces are not only congruent but equivalent in that there is a symmetry of the polyhedron which transforms one face into the other; and, in the same sense, any two vertices of the polyhedron are equivalent. Each vertex is regular, in the sense that the angles meeting at the vertex are all equal, and the dihedral angles of the edges ending at the vertex are all equal. There are five Platonic polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.

One way to generalize the Platonic polyhedra results in the convex **uniform polyhedra**. A uniform polyhedron has regular polygons as faces, and any two vertices are equivalent under some symmetry of the polyhedron. The faces need not all be congruent to one another. There are two infinite families of convex uniform polyhedra, the prisms and the antiprisms; there are also thirteen others, called **Archimedean** polyhedra, that are not Platonic.[1]

There are **uniform** polyhedra that are not convex. The major reference for these is [6]; the enumeration in that paper was proved to be complete in [23].

Another generalization of the Platonic polyhedra leads to the **dual uniform polyhedra**. In a dual uniform polyhedron, each vertex is regular, in the sense given above; the faces need not be regular, but any two faces are equivalent under a

[1]Clearly, every Platonic polyhedron satisfies the conditions for being uniform.

symmetry of the polyhedron. Besides the Platonic polyhedra, there are two infinite families of convex dual uniform polyhedra, the dipyramids and the trapezohedra, and thirteen others, the **Catalan** polyhedra.[2]

Descriptions of these polyhedra, with advice on how to build models, can be found in many places. The book by Cundy and Rollett[9] is this author's favorite. Wenninger's books, [24] and [25], are also very good. Some interesting history may be found in [12]. See also [13], for colorful illustrations which take the concept of stellation in directions not explored here.

One may notice several correspondences between the uniform and the dual uniform polyhedra as they have just been described. Indeed, the correspondence is close; Catalan polyhedra are often called "Archimedean dual" polyhedra. "Duality" is based on a geometric concept called "polarity," which will be explained in §2.3. One may also consult [25].

The application of computer graphics to the depiction of stellated polyhedra is an attractive project, on which many people have worked. For a single base polyhedron, analysis of its possible stellations can be carried out *ad hoc*; but to cope with the more complex cases, and to study a large class of base polyhedra, it seems better to perform the analysis, from first principles, by computer. How to do so is the subject of the present paper.

1.2. **Rules for Stellation.** The classic work on stellations is the little monograph [7], recently reprinted [8], which completely describes the stellations of the icosahedron. This work also gives a precise definition of "stellation," ascribed to J. C. P. Miller, which we formulate as follows.

Let $\mathcal{P}$ be a polyhedron, and let $\mathbf{G}$ be a group of symmetries of $\mathcal{P}$. The faces of $\mathcal{P}$ lie on a set of **facial planes**. Each facial plane is divided, by its intersections with the other facial planes, into closed convex **facets**, which may be compact or non-compact. Also, 3-space is divided by the facial planes into closed convex **cells**, which also may be compact or non-compact. The **boundary** of a cell is made up of facets. A polyhedron S is a **stellation** of $\mathcal{P}$ for the group $\mathbf{G}$ if

(1) S is a union of some of the cells determined by $\mathcal{P}$;
(2) S is symmetric under $\mathbf{G}$;
(3) every component of the complement of S is non-compact;
(4) it is not possible to partition S into two polyhedra satisfying conditions (1)-(3) and connected at most along edges.

These four conditions deserve some amplification. Condition (1) allows both compact and non-compact cells; [7] considers only stellations made of compact cells. The group $\mathbf{G}$ may be the entire group of symmetries of $\mathcal{P}$, or a subgroup. When the symmetry group of $\mathcal{P}$ contains both rotations and reflections, $\mathbf{G}$ is often taken to be the **general group** containing both rotations and reflections, or the **special group** containing only the rotations.

We shall say that a stellation is **reversible** if it is symmetric under the general group; if it is symmetric only under the special group, we shall say that it is **chiral**. In the latter case, the stellation and its mirror image are also said to be **enantiomorphous**.

Condition (3) can be stated more simply if only compact stellations are considered; then it requires that the complement of S be connected. In the general case,

[2]Eugène Charles Catalan described them in [2].

it excludes uninteresting possibilities such as the closure of the complement of a compact stellation. Because a stellation is a union of closed cells, condition (3) also excludes such cases as a hollow shell of cells that are connected to one another along edges.

2. It's All About the Symmetry Group

Whatever finite polyhedron is chosen for the base, it will have a finite symmetry group, consisting of orthogonal transformations of Euclidean space. There will be a point left invariant by all elements of the group; this is the **center** of the polyhedron; we will assume that this is the origin of coordinates. The symmetry group of a Platonic, uniform, or dual uniform polyhedron is, in most cases, generated by reflections. Such a group is called a **reflection group**. For two Archimedean and two Catalan polyhedra, the symmetry group is the subgroup of a reflection group consisting of rotations.

Any reflection in three-dimensional space determines, and is determined by, a plane. The planes of the reflections in a reflection group divide the unit sphere around the origin into congruent regions; this tessellation of the sphere is the **kaleidoscope** of the group.

It is possible, and enlightening, to start with the group and derive the base polyhedron from it, rather than the other way round. Reflection groups have been studied and classified. In each case of interest, three generators R_1, R_2, and R_3 suffice. Because these are reflections, they have order 2:

$$\mathsf{R}_i^2 = \mathsf{E}\,. \tag{2.1}$$

Because the group is finite, each product of two generators has finite order:

$$(\mathsf{R}_i\mathsf{R}_j)^{m_{ij}} = \mathsf{E}\,, \quad i \neq j\,. \tag{2.2}$$

And that's it. The relations (2.1) and (2.2) completely determine a group in which, if R_i is the reflection in the plane P_i, then P_i and P_j meet at an angle of π/m_{ij}. Thus, the information contained in (2.1) and (2.2) suffices to define the geometry of the reflection group. It is clear that the reflections in the planes P_i satisfy those relations. What is not obvious, and is in fact a fairly deep result, is that every relation in the group generated by those reflections is a consequence of (2.1) and (2.2); see Humphreys[14].

In the following parts of this section, we first describe the spherical reflection groups, and their kaleidoscopes, more fully. We follow Conway and Smith[3] for the names of the groups. Next we show how the Platonic and convex uniform polyhedra can be constructed from the spherical kaleidoscopes. The next subsection explains the "polarity" or **correlation** associated with a sphere, and uses it to derive the dual uniform polyhedra from the uniform ones. Finally, we define the **constellation** of a symmetrical polyhedron, from which stellations can be constructed; and we show how the construction of a constellation may be facilitated by group-theoretical techniques.

2.1. **Finite Reflection Groups.** A finite reflection group in three dimensions determines, and is determined by, the spherical triangle which the planes of the generators R_i define on the unit sphere. Let us call the angles of this triangle $(\pi/p,\ \pi/q,\ \pi/r)$ for some integers (p, q, r). The sum of the angles must be greater than π, so $1/p + 1/q + 1/r > 1$. These are the possible values of (p, q, r):

FIGURE 1. The prismatic tessellation with $n = 5$

(1) $(2, 2, n)$ where $n \geq 2$. The group is of order $4n$, and is called the n-**gonal holo-prismatic group**. The tessellation is shown in Fig. 1, for $n = 5$.
(2) $(2, 3, 3)$. The group is of order 24, and is called the **holo-tetrahedral group**. The tessellation is shown in Fig. 2.
(3) $(2, 3, 4)$. The group is of order 48, and is called the **holo-octahedral group**. The tessellation is shown in Fig. 3.
(4) $(2, 3, 5)$. The group is of order 120, and is called the **holo-icosahedral group**. The tessellation is shown in Fig. 4.

There are other finite groups of orthogonal transformations. In particular, the **rotational subgroup** of a reflection group is of index 2, containing those elements of the group that are rotations. The names of the rotational subgroups have the prefix **chiro-** in place of "holo-".

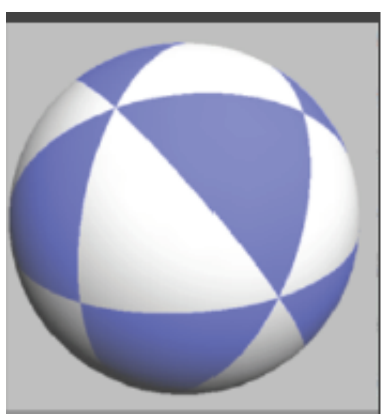

FIGURE 2. The tetrahedral tessellation

2.2. **Wythoff Construction of Platonic and Convex Uniform Polyhedra.** For a reflection group **G**, we may construct a polyhedron with the symmetry of **G**

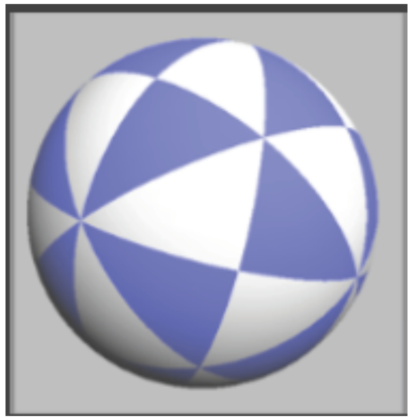

FIGURE 3. The octahedral tessellation

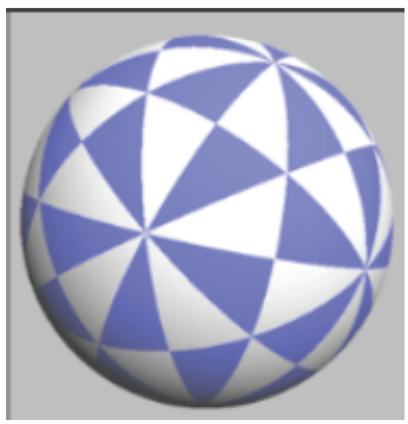

FIGURE 4. The icosahedral tessellation

as follows. Choose a point V; find the **orbit**[3] of V, i.e. the set of all images of V under elements of $\mathbf{G}$; let the convex hull of this orbit be the polyhedron. For certain choices of V, the polyhedron will be Platonic or Archimedean. Coxeter *et al.* [6] have described a systematic method for choosing such points V.

Let us denote by PQR the spherical triangle, on a sphere centered at O, determined by the kaleidoscope group, where the angles at the vertices (P, Q, R) are respectively (π/p, π/q, π/r). When we need to refer to the reflections in the planes determined by the sides of this triangle, we shall denote by R_{pq} the reflection in the plane of side PQ, and similarly for R_{qr} and R_{rp}. Figure 5 shows seven of the eight possible choices for the case of $(2, 3, 5)$. A helpful reference for locating the points in this figure is [19]. I hope to add an interactive panel, capable of showing

[3] This use of the word "orbit" may seem bizarre when it is first encountered. The earliest examples of this usage that I know of are in studies of the effects of iterating a single transformation T; one application of T is analogous to one step forward in a dynamical system with discrete time.

more information, to my web site at `http://mathinteract.com/Kaleidoscopes/UniformTessellations.html`.

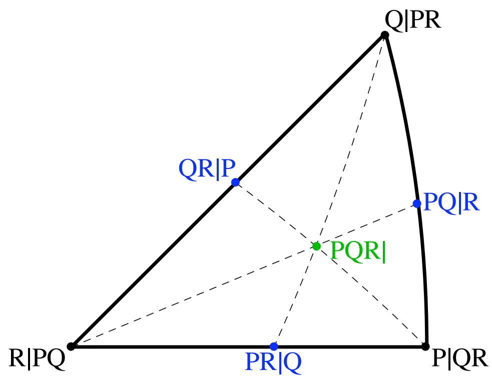

FIGURE 5. Wythoff Construction, non-snub cases

2.2.1. *The Regular and Quasi-Regular Cases.* Let V be a vertex of the spherical triangle, for instance P. A symbol for this choice is $p|qr$. Around the vertex Q, there are points in the orbit of P which form the vertices of a polygon of q sides, which we denote by $\{q\}$. Similarly, around the vertex R are the vertices of a polygon $\{r\}$. At the vertex P, the faces $(\{q\}, \{r\})$ are repeated p times.

If q or r equals 2, then the polygon $\{2\}$ collapses into a single edge. Therefore, the symbol $p|2r$ determines a polyhedron in which p faces of type $\{r\}$ surround each vertex; this is a Platonic polyhedron, for which the Schläfli symbol is $\{r, p\}$.

If p equals 2, then a vertex is surrounded by faces of types $(\{q\}, \{r\}, \{q\}, \{r\})$. Coxeter[4] has extended Schläfli's notation to all the convex uniform polyhedra, and this one has the symbol $\left\{ {q \atop r} \right\}$.

The holo-prismatic group gives some less interesting cases. $2|2r$ is the Wythoff symbol for a "dihedron," whose Coxeter symbol is $\{r, 2\}$; it consists of two faces $\{r\}$ back to back. $p|22$ is the Wythoff symbol for a bundle of p digons; the Coxeter symbol would be $\{2, p\}$.

2.2.2. *The Semi-Regular Cases.* If V is on one side of the spherical triangle, say PQ, then its orbit includes the vertices of a regular $\{p\}$ surrounding P and a regular $\{q\}$ surrounding Q. Surrounding R there are the vertices of a $2r$-sided polygon, which will be regular if V is equidistant from PR and QR. This is the case, when V is the intersection of PQ with the bisector of the angle at R. The Wythoff symbol for the resulting polyhedron is $pq|r$; each vertex is surrounded by polygons of type $\{p\}$, $\{2r\}$, $\{q\}$, $\{2r\}$.

If p or q equals 2, then the corresponding polygon collapses into a single edge. The polyhedron $2q|r$ has faces of type $\{q\}$, $\{2r\}$, $\{2r\}$ at a vertex. It is called the "truncated $\{r, q\}$," because it can be constructed from $\{r, q\}$ by cutting off a little pyramid at each vertex. The Coxeter symbol is $\mathrm{t}\{r, q\}$.

If r equals 2, then the polyhedron $pq|2$ has faces of type $\{p\}$, $\{4\}$, $\{q\}$, $\{4\}$ at a vertex. It is called the "rhombic $\left\{ {p \atop q} \right\}$," and its Coxeter symbol is $\mathrm{r}\left\{ {p \atop q} \right\}$.

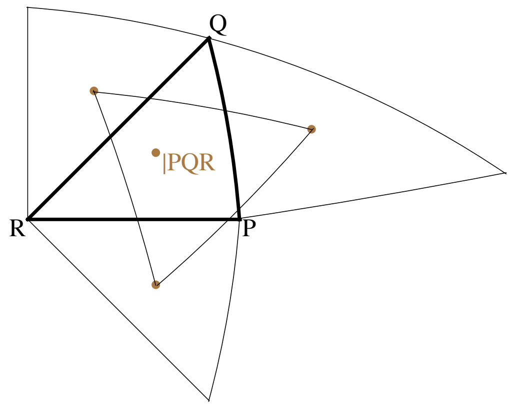

FIGURE 6. Wythoff Construction, snub case

The case $2q|2$ could be regarded as a "truncated $\{2, q\}$" or a "rhombic $\{q, 2\}$." Either way, it has polygons of type $\{q\}$, $\{4\}$, $\{4\}$ at a vertex, and is known as the "q-gonal prism."

2.2.3. *The Even-Faced Cases.* If V is interior to the spherical triangle, then its orbit defines polygons with $2p$, $2q$, and $2r$ vertices around P, Q, and R respectively. These will be regular polygons if V is at the *incenter* of the triangle, where it is equidistant from all three sides. The Wythoff symbol is $pqr|$. The polyhedron $2qr|$ is traditionally known as the "truncated $\left\{ {q \atop r} \right\}$," with Coxeter symbol $\mathrm{t}\left\{ {q \atop r} \right\}$. However, this term must be interpreted with some latitude, because truncating $\left\{ {q \atop r} \right\}$ would result in a polyhedron with rectangular, not square, faces. For this reason, some authors modify the name; for instance, "rhombitruncated $\left\{ {q \atop r} \right\}$" in reference [24].

In the case of the holo-prismatic group, this construction gives a polyhedron with additional symmetry: $22r|$ is a $2r$-gonal prism, equivalent to $2q'|2$ where $q' = 2r$.

2.2.4. *The Snub Cases.* There is one more way to construct an Archimedean polyhedron from a reflection group. With respect to the special subgroup, the images of the spherical triangle PQR split into two orbits, which are shown in white and blue in Figs. 1-4. Each spherical triangle is surrounded by three triangles belonging to the other orbit; they are mirror images of the original triangle in its three sides. If V is a point interior to the spherical triangle, consider its mirror images and the orbit, with respect to the restricted group, which contains them. This orbit includes points which surround P, Q, and R with polygons of type $\{p\}$, $\{q\}$, and $\{r\}$ respectively. The other faces determined by this orbit are triangles, which in general are scalene. But there is one choice of V for which these triangles are equilateral. An example is shown in Figure 6. The Wythoff symbol for the polyhedron is $|pqr$.

The polyhedron $|2qr$ could be called the "snub $\left\{ {q \atop r} \right\}$," with Coxeter symbol $\mathrm{s}\left\{ {q \atop r} \right\}$. (In point of fact, the names "snub cube" instead of "snub cuboctahedron," and "snub dodecahedron," are standard.) The snub form of the dihedron, with Wythoff symbol $|22r$, is called the "r-gonal antiprism;" its faces include two parallel $\{r\}$'s, one rotated with respect to the other through an angle of π/r, and $2r$ triangles, one

for each edge of each $\{r\}$; the triangles share common edges that run in a zigzag between the two $\{r\}$'s.

The r-gonal antiprism $|22r$ has a larger symmetry group than one might expect. The r-gonal chiro-pyramidal group is of order $2r$; the $2r$-gonal holo-pyramidal group[4] is of order $8r$. Intermediate between these groups is the r-gonal **pro-antiprismatic** group, of order $4r$. See Conway and Smith[3] for details.

2.2.5. *Summary of the Wythoff Construction.* The Wythoff construction of the Platonic and Archimedean polyhedra is attractive because it is a unified theory of what would otherwise be a more diverse set of entities. However, the correspondence between Wythoff symbols and polyhedra is not completely simple. It includes some "degenerate cases," in particular the dihedra $2|2r$ and the "bundles" $p|22$. Also, there are several cases in which two or more Wythoff symbols determine the same polyhedron. The most striking of these is the "snub tetrahedron," $|233$, which turns out to be none other than an icosahedron.

The convex uniform polyhedra, with their usual Wythoff and Coxeter symbols, are listed in Table 1.

2.3. **Polarity and Duality.** Associated with any sphere $\mathcal{S}$ in three-dimensional space is a **polarity** $\mathcal{P}_{\mathcal{S}}$, which makes a point correspond not to another point but to

[4] As we saw in §2.2.3, this is the symmetry group of $22r|$.

	Wythoff	Coxeter	name
Platonic	$3\|23$	$\{3,3\}$	tetrahedron
	$3\|24$	$\{4,3\}$	cube = hexahedron
	$4\|23$	$\{3,4\}$	octahedron
	$3\|25$	$\{5,3\}$	dodecahedron
	$5\|23$	$\{3,5\}$	icosahedron
Archimedean	$2\|34$	$\left\{\begin{smallmatrix}3\\4\end{smallmatrix}\right\}$	cuboctahedron
	$2\|35$	$\left\{\begin{smallmatrix}3\\5\end{smallmatrix}\right\}$	icosidodecahedron
	$23\|3$	$\mathrm{t}\{3,3\}$	truncated tetrahedron
	$23\|4$	$\mathrm{t}\{4,3\}$	truncated cube
	$24\|3$	$\mathrm{t}\{3,4\}$	truncated octahedron
	$23\|5$	$\mathrm{t}\{5,3\}$	truncated dodecahedron
	$25\|3$	$\mathrm{t}\{3,5\}$	truncated icosahedron
	$34\|2$	$\mathrm{r}\left\{\begin{smallmatrix}3\\4\end{smallmatrix}\right\}$	rhombicuboctahedron
	$35\|2$	$\mathrm{r}\left\{\begin{smallmatrix}3\\5\end{smallmatrix}\right\}$	rhombicosidodecahedron
	$234\|$	$\mathrm{t}\left\{\begin{smallmatrix}3\\4\end{smallmatrix}\right\}$	truncated cuboctahedron
	$235\|$	$\mathrm{t}\left\{\begin{smallmatrix}3\\5\end{smallmatrix}\right\}$	truncated icosidodecahedron
	$\|234$	$\mathrm{s}\left\{\begin{smallmatrix}3\\4\end{smallmatrix}\right\}$	snub cube
	$\|235$	$\mathrm{s}\left\{\begin{smallmatrix}3\\5\end{smallmatrix}\right\}$	snub dodecahedron
infinite families	$2q\|2$	$\mathrm{t}\{2,q\}$	q-gonal prism
	$\|22r$	$\mathrm{s}\left\{\begin{smallmatrix}2\\r\end{smallmatrix}\right\}$	r-gonal antiprism

TABLE 1. Platonic and Archimedean polyhedra

a plane, considered as a single entity. At the same time, $\mathcal{P}_\mathcal{S}$ makes a line correspond to another line and a plane to a point.

Polarity is different from inversion in a sphere; but it is convenient to define inversion first. Let $\mathcal{S}$ have center C and radius r. If P is a point different from C, then the **inverse** of P with respect to $\mathcal{S}$ is the point Q on the half-line starting from C and passing through P, such that $\overline{CP} \cdot \overline{CQ} = r^2$.

Now $\mathcal{P}_\mathcal{S}$ is defined as follows for a point P different from C, a line L that does not pass through C, and a plane $\mathfrak{p}$ that does not pass through C.

- Let P' be the inverse of P with respect to $\mathcal{S}$; then $\mathcal{P}_\mathcal{S}P$ is the plane through P' perpendicular to CP'. $\mathcal{P}_\mathcal{S}P$ is called the **polar** of P with respect to $\mathcal{S}$.
- Let M be the point on L such that CM is perpendicular to L, and let M' be the inverse of M with respect to $\mathcal{S}$; then $\mathcal{P}_\mathcal{S}\mathsf{L}$ is the line through M' perpendicular to the plane of C and L. $\mathcal{P}_\mathcal{S}\mathsf{L}$ is called the **polar line** of L with respect to $\mathcal{S}$.
- Let P_0 be the point on $\mathfrak{p}$ such that CP_0 is perpendicular to $\mathfrak{p}$, and let P be the inverse of P_0 with respect to $\mathcal{S}$; then $\mathcal{P}_\mathcal{S}\mathfrak{p} = P$. P is called the **pole** of $\mathfrak{p}$ with respect to $\mathcal{S}$.

Among the properties of $\mathcal{P}_\mathcal{S}$ are these:

- Let $\mathbf{X}$ be any geometrical element—point, line, or plane—for which $\mathcal{P}_\mathcal{S}\mathbf{X}$ has been defined. Then $\mathcal{P}_\mathcal{S}(\mathcal{P}_\mathcal{S}\mathbf{X}) = \mathbf{X}$.
- Let $\mathbf{X}$ and $\mathbf{Y}$ be geometrical elements which are in the domain of $\mathcal{P}_\mathcal{S}$, and **incident**, i.e., one is contained in the other. Then $\mathcal{P}_\mathcal{S}\mathbf{X}$ and $\mathcal{P}_\mathcal{S}\mathbf{Y}$ are incident.
- A point P is incident with $\mathcal{P}_\mathcal{S}P$ if and only if P lies on $\mathcal{S}$. In this case, $\mathcal{P}_\mathcal{S}P$ is the plane tangent to $\mathcal{S}$ at P.
- A line L is incident with $\mathcal{P}_\mathcal{S}\mathsf{L}$ if and only if L is tangent to $\mathcal{S}$. In this case, let P be the point of contact of L with $\mathcal{S}$, and let $\mathfrak{p} = \mathcal{P}_\mathcal{S}P$; then $\mathcal{P}_\mathcal{S}\mathsf{L}$ is the line in $\mathfrak{p}$, perpendicular to L at P.
- If the point P is outside the sphere $\mathcal{S}$, then lines drawn through P tangent to $\mathcal{S}$ meet the sphere at the points of a small circle, which lies in the plane $\mathcal{P}_\mathcal{S}P$.

Now, a polyhedron has vertices, which are points; and edges, which are segments on certain lines, and faces, which are figures on certain planes. The incidence relations of these points, lines, and planes are a sort of abstract description of the polyhedron. Note that each edge is on a line which is incident with two vertices, and is also incident with two facial planes. We get the **dual** of the polyhedron, with respect to the sphere $\mathcal{S}$, by applying $\mathcal{P}_\mathcal{S}$ to these points, lines, and planes: the poles of the facial planes of the original polyhedron are the vertices of the dual polyhedron; the edges of the dual polyhedron are segments on the polar lines; and the faces of the dual are on the polar planes of the vertices of the original.

The relation of duality among polyhedra is symmetrical; that is, if $\mathcal{Q}$ is the dual of $\mathcal{P}$, then $\mathcal{P}$ is the dual of $\mathcal{Q}$. Sometimes it is convenient to give priority to one member of this pair, and to say that it is the **primal** polyhedron, while the other is the dual one.

Let us consider applying this construction to a Platonic or convex uniform polyhedron. Wenninger [25] has pictures of examples. The polyhedron's vertices lie on a sphere; let its radius be R. The edges are all of the same length, which we will

denote $2l$. It can be seen that the midpoints of the edges lie on a sphere of radius $\sqrt{R^2 - l^2}$. Let $\mathcal{S}$ be this sphere.

The construction of the dual polyhedron with respect to this $\mathcal{S}$ may be more easily visualized if we take it one face at a time. Let V_0 be a vertex of the original polyhedron. The edges arising from V_0 all have their midpoints on a single plane $\mathfrak{p}_0$, and in fact on a circle in that plane. The polygon made by joining these midpoints is called the **vertex figure** of the original polyhedron.

The plane which contains that vertex figure and its circumscribed circle is the polar of P_0. The polar lines of those edges lie in that plane, and are tangent to that circle. Now, if two successive edges arising from V_0 are on lines L_1 and L_2, then they lie on the plane of a face of the original polyhedron; the pole of this plane is the intersection of $\mathcal{P}_{\mathcal{S}}\mathsf{L}_1$ and $\mathcal{P}_{\mathcal{S}}\mathsf{L}_2$. And this pole is a vertex of the dual polyhedron. By this reasoning, we can construct a typical face of the dual. We start with a vertex figure of the original polyhedron, and its circumscribed circle. At each corner of the vertex figure, we draw a line tangent to the circle. These lines are the edges of the face of the dual polyhedron. This construction is ascribed to Dorman Luke (see [25] or [9]).

The dual of a Platonic polyhedron is a Platonic polyhedron. A vertex with p edges is dual to a plane on which p lines lie; and symmetrically, of course, a face plane with q edges is dual to a vertex where q edges meet. In short, the dual of $\{p, q\}$ is $\{q, p\}$. In particular, the dual of a tetrahedron is another tetrahedron, but the vertices of the dual correspond to the faces of the original.

The dual of an Archimedean polyhedron is a Catalan polyhedron. The dual of a prism is a dipyramid, and that of an antiprism is a trapezohedron. Thus generally the dual of a convex uniform polyhedron is—unsurprisingly—what we have been calling a convex dual uniform polyhedron.‘ In this relationship we shall take the uniform polyhedron to be “primal.”

Let us introduce the notation $D(\cdots)$ to denote the dual of a Wythoff symbol. The dual uniform polyhedra, with names as in [25], are listed in Table 2.

	symbol	name
Catalan	$D(2\|34)$	rhombic dodecahedron
	$D(2\|35)$	rhombic triacontahedron
	$D(23\|3)$	truncated tetrahedron
	$D(23\|4)$	triakis octahedron
	$D(24\|3)$	tetrakis hexahedron
	$D(23\|5)$	triakis icosahedron
	$D(25\|3)$	pentakis dodecahedron
	$D(34\|2)$	deltoidal icositetrahedron
	$D(35\|2)$	deltoidal hexecontahedron
	$D(234\|)$	disdyakis dodecahedron
	$D(235\|)$	disdyakis triacontahedron
	$D(\|234)$	pentagonal icositetrahedron
	$D(\|235)$	pentagonal hexecontahedron
infinite families	$D(2q\|2)$	q-gonal dipyramid
	$D(\|22r)$	r-gonal trapezohedron

TABLE 2. Dual uniform polyhedra

2.4. **Summary of Following Sections.** The rest of this paper describes how a computer program can describe the stellations of a Platonic, convex uniform, or convex dual uniform polyhedron $\mathcal{P}$, starting from its Wythoff (or dual Wythoff) symbol. In §3, we consider some of the geometrical elements of this description, in particular how to describe the planes, the **edges** defined by the intersections of pairs of planes, and the **vertices** in which three or more planes intersect.

The description of stellations depends on incorporating the edges and vertices into a combinatorial structure, which we call the **constellation** of the polyhedron. Further ingredients of this structure are explained in §4. Some of them are systematically labelled, as we explain in §5.

Much of the structure of the constellation is expressed by the **cell orbit graph**, which is an abstract graph, having labels on both nodes and edges. It is introduced in §6. The stellations of $\mathcal{P}$ are in one-to-one correspondence with certain subsets of the nodes of this graph. From such a subset one may read off the facets which form the boundary of the stellation. We will characterize the subsets that correspond to stellations in §6.2.

How many stellations are there, for a given $\mathcal{P}$? This is a well-established question—consider the title of [7]. We can answer it if we can count the appropriate sets of nodes in the cell orbit graph. In §8, we describe a counting algorithm, and in §8.4 we discuss some of the results.

3. Geometrical Elements

3.1. **Determination of the Facial Planes.** The facial planes of the polyhedron $\mathcal{P}$ are fundamental to determining the stellations of $\mathcal{P}$. If $\mathcal{P}$ is determined by a (non-dual) Wythoff symbol, then §2.2 shows how to determine a vertex V. In all cases except the snub polyhedra, every orbit of faces contains at least one for which V is a vertex; the other vertices of this face are obtained by rotating V around the axis through one of the vertices, such as P, of the spherical triangle. Then the facial plane is just the plane through V perpendicular to OP.

In a snub polyhedron, the construction is different, as depicted in figure 6. If $p > 2$ then there is a face of type $\{p\}$ surrounding the axis OP and having $\mathsf{R}_{\mathsf{pq}}V$ as a vertex; its facial plane is perpendicular to OP and passes through $\mathsf{R}_{\mathsf{pq}}V$. There may similarly be facial planes perpendicular to OQ and OR. Finally, the plane of the triangular "snub" face is determined by $\mathsf{R}_{\mathsf{pq}}V$, $\mathsf{R}_{\mathsf{qr}}V$, and $\mathsf{R}_{\mathsf{rp}}V$.

For the dual Wythoff symbol the determination of a facial plane is simpler. It is just the plane through the vertex V and perpendicular to OP.

3.2. **Orientation Is Important.** It is helpful to use **oriented** lines and planes in exploring the geometry of stellations of a polyhedron. For a line, the orientation selects a preferred direction. For a plane, it selects a preferred sense of rotation on the plane, and a preferred direction on the normal line. From the side of the plane pointed to by the positive direction of the normal, the positive direction of rotation is counterclockwise.

The boundary of a cell consists of facets, which are assumed to be oriented with the normal line pointing *out* of the cell. Each facet has a boundary also; consisting of line segments going around the facet in the preferred direction of rotation.

The oriented intersection of two oriented planes is dependent on the order in which the two planes are taken. Let P_1 and P_2 be oriented planes with normal

vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively. The oriented intersection has positive direction $\mathbf{v}$ such that $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{v})$ is a right-handed triple.

For every oriented element in the constellation, there is an element with the same position but opposite orientation. This duplication of data makes some details easier to understand, or to implement. For instance, when two cells are adjacent, the boundary of one has a facet which is the opposite of a facet in the boundary of the other. We may say that an oriented plane or facet is **positive** if the normal vector points away from the center of the polyhedron, and **negative** otherwise.[5] Then it becomes easier to speak of assigning different colors to the positive and negative facets of a stellation.

3.3. **Plücker Coordinates.** What sort of data structure is useful for specifying a plane or a line in $\mathbb{R}^3$? We will use a version of Plücker coordinates.[6]

A plane is the locus of points $\mathbf{p}$ such that $\mathbf{n} \cdot \mathbf{p} = d$ for some vector $\mathbf{n}$ and some real number d. The vector $\mathbf{n}$ is normal to the plane. Together, $\mathbf{n}$ and d serve as **homogeneous Plücker coordinates** for the plane; they are homogeneous because they can be multiplied by a non-zero factor without affecting the plane they specify. For purposes of computation, it is more convenient to have a one-to-one relation between the geometrical entity and the descriptive data. We can achieve that by stipulating that the vector $\mathbf{n}$ is of unit length. At first sight this seems to be ambiguous; the pair $(\mathbf{n}, d)$ could be replaced by the pair $(-\mathbf{n}, -d)$. But this turns out to our advantage—these two sets of coordinates specify two oriented planes. We make the convention that $\mathbf{n}$ is the unit normal in the positive direction, to define the **normalized Plücker coordinates** for an oriented plane. Note that the plane is positive, in the sense defined in §3.2, if $d > 0$ and negative if $d < 0$.

A line is the locus of points $\mathbf{p}$ such that $\mathbf{p} \times \mathbf{b} = \mathbf{t}$, where $\mathbf{b}$ and $\mathbf{t}$ are vectors such that $\mathbf{b} \neq 0$ and $\mathbf{b} \cdot \mathbf{t} = 0$. The vector $\mathbf{b}$ is parallel to the direction of the line. As before, the pair $(\mathbf{b}, \mathbf{t})$ are homogeneous Plücker coordinates. If we stipulate that $||\mathbf{b}|| = 1$, and that $\mathbf{b}$ determines the positive direction on the line, then we have normalized Plücker coordinates for an oriented line.

4. The Constellation

If $\mathcal{P}$ is a polyhedron, with a symmetry group $\mathbf{G}$, then the **constellation** of $(\mathcal{P}, \mathbf{G})$ consists of the ingredients of the stellations of $\mathcal{P}$. These include the cells into which space is divided by the planes of $\mathcal{P}$; the facets on the boundaries of those cells;the segments on the boundaries of those facets;and the edges and vertices which define those facets. The action of $\mathbf{G}$ on all these things is also part of the picture. Facet orbits and cell orbits will be defined using the special subgroup of $\mathbf{G}$. If $\mathbf{G}$ also contains reflections, then these induce a transformation of an orbit into its **mirror image**, which may or may not be distinct from it. A stellation may be chosen by choosing a set of cell orbits. Finally, there is a labelled graph, the **cell orbit graph**, which can be used to characterize those sets of cell orbits which we accept as legitimate stellations.

How shall the data representing these things, and the logical relations among them, be defined and computed?

[5] For the Platonic, Archimedean, and Catalan polyhedra, no facial planes pass through the center of the polyhedron. Some non-convex uniform polyhedra[6] do have such facial planes.

[6] This is the tip of a large iceberg, generally called Grassmann-Plücker coordinates, for specifying a k-dimensional subspace of n-dimensional projective space.

4.1. **Facial Planes, and the Symmetry Group.** Constructing the constellation starts with the set of face planes. In §3.1 we explained how to find a facial plane for each orbit of faces. Actually, as stated above (§3.2), it is useful to let the facial planes be oriented planes, and define two opposite planes for each face of $\mathcal{P}$.

How about the other members of the orbit? Each facial plane has an **isotropy subgroup**, which is the set of elements of $\mathbf{G}$ which move it into itself; call this subgroup $\mathbf{H}$. The **left cosets** of $\mathbf{H}$, that is, the sets of the form $\mathsf{g}\mathbf{H}$ where $\mathsf{g} \in \mathbf{G}$, are in a natural correspondence with the facial planes of that orbit.

The set of left cosets of $\mathbf{H}$ is a **homogeneous space**, and is denoted by $\mathbf{G}/\mathbf{H}$. There is a natural action of $\mathbf{G}$ on $\mathbf{G}/\mathbf{H}$; under the correspondence with facial planes, this action translates into the correct action of $\mathbf{G}$ on this orbit of facial planes.

4.2. **Structures on a Facial Plane.** When the facial planes of $\mathcal{P}$ have been determined, the other entities in the constellation can be derived from them. Many of these entities lie within a particular facial plane.

The collection of these is called a **stellation diagram** by authors such as Wenninger [24], [25].

4.2.1. *Edge.* An **edge** may be defined as the oriented intersection of two facial planes of $\mathcal{P}$. For each edge that lies on a particular facial plane, the opposite edge is also constructed. If P_1 and P_2 are two non-parallel oriented facial planes of $\mathcal{P}$, then the edge they determine will be called $\mathrm{Edge}(\mathsf{P}_1, \mathsf{P}_2)$.

An edge E lying on an oriented plane P is **positive** for P if the Plücker coordinates of E are $(\mathbf{b}, \mathbf{t})$, those of P are $(\mathbf{n}, d)$, and $\mathbf{t} \cdot \mathbf{n} > 0$. Intuitively, this means that the direction of E is counter-clockwise, with respect to the orientation of P, as seen from the point on P closest to the origin.

4.2.2. *Vertex.* When three or more facial planes of $\mathcal{P}$ intersect in a point, that point is a **vertex** of the constellation of $\mathcal{P}$. The vertices that lie on a particular facial plane may also be defined as the intersections of two or more edges in that plane. Even without the presence of opposite oriented planes in the constellation, there may be, in general, more than one triple of planes that intersect in one point. In a computer implementation that uses floating-point numbers for the coordinates of points, care must be taken to identify points whose coordinates differ only in round-off errors.[7]

4.2.3. *Segment.* Each edge is divided into **segments** by the vertices on the edge. A segment inherits an orientation from the enclosing edge, and so has a first and a last vertex. We distinguish compact from non-compact segments, by giving a non-compact segment an "infinite vertex." Sufficient information about the infinite vertex is gained from the vector pointing forward along the enclosing oriented edge.

4.2.4. *Facet.* An oriented facet is constructed by finding the list of oriented segments making up its boundary. Every oriented segment that lies on a facial plane is used, exactly once, in the boundary of an oriented facet. The boundaries can be constructed if we know, for each oriented segment, the next one belonging to the same facet.

[7] I think that the resulting code is a blemish upon my program. The coordinates are all algebraic numbers, and in principle it is possible to determine with exactitude which triples of planes intersect in the same point. But the challenge of doing so is formidable.

There is an elegant relation which determines the map from a segment to its successor. Indeed, this hold true for any planar map, and is an important step in recent proofs of the Four-Color Theorem [11]. Let $\mathfrak{S}$ be the set of oriented segments making up the map. Define three functions, $\mathfrak{e}$, $\mathfrak{n}$, and $\mathfrak{f}$: $\mathfrak{S} \to \mathfrak{S}$ as follows:

- $\mathfrak{e}$ sends each oriented segment to its opposite;
- at every vertex v, $\mathfrak{n}$ permutes the segments whose first vertex is v by sending each one to the next-door neighbor in a counterclockwise direction;
- in every facet F, $\mathfrak{f}$ permutes the segments on the boundary of F, by sending each one to its successor in a counterclockwise direction.

Then

$$\mathfrak{f} \circ \mathfrak{e} \circ \mathfrak{n} = \mathrm{id}\,.$$

Strictly speaking, this equation applies only to a planar map that is confined to a bounded part of the plane. For us, it is possible in principle to add a "point at infinity," to be the first point of the segments that have no finite first point. In practice, it is easier to exclude those segments from the domain, and also the range, of $\mathfrak{n}$, and take the above equation in the form

$$\mathfrak{f} = \mathfrak{n}^{-1} \circ \mathfrak{e}$$

(note that $\mathfrak{e}^{-1} = \mathfrak{e}$). The effect is that $\mathfrak{f}$ is not defined for segments whose last point is infinite.

If a facet F lies on the facial plane P, let $\mathbf{p}_0$ be the point on P closest to the origin; then the **power** of F is defined as the number of positive edges intersected by a line segment from $\mathbf{p}_0$ to a point in the interior of F. The set of all facets on P with a given power is called a **stratum**.

4.2.5. *Facet Orbit.* The isotropy group $\mathbf{H}$ of a face is also a symmetry group for the stellation diagram on that facial plane. An orbit under $\mathbf{H}$ of facets on this plane is part of an orbit under $\mathbf{G}$ of the set of all facets of the stellation. Therefore there is no real need to compute the facets anew on every facial plane in the constellation; each one is simply the image of one on the original facial plane, under an element of the symmetry group. For the more complicated constellations, this is an important optimization.

The **power** of a facet orbit is the power of any one of its facets. A **stratum** of facet orbits is the set of facet orbits of a facial plane with a given power. Equivalently. it is the set of orbits of facets belonging to a stratum of some facial plane.

4.3. **Cells.** A cell is found by constructing its boundary, which is a set of oriented facets. An oriented facet f_0 connects to other oriented facets on the boundary of the same cell, along the segments making up the boundary of f_0 . Let s be such a segment; of P_0 is the oriented facial plane containing f_0, then s is a segment on $\mathrm{Edge}(\mathsf{P}_0, \mathsf{P}_1)$ for some oriented facial plane P_1 . Let s' be the opposite segment to s; it is contained in $\mathrm{Edge}(\mathsf{P}_1, \mathsf{P}_0)$. The facet f_1 on P_1 containing s' is on the boundary of the same cell as f_0.

The set of all the facets in the constellation of $\mathcal{P}$ may be treated as the nodes of a graph, in which there is an arc connecting any two facets that are related as are f_0 and f_1 in the preceding paragraph. Then the connected components of this graph are the boundaries of the cells of the constellation.

The **power** of a cell is defined by drawing a line segment from the interior of the cell to the center; the power is then the number of positive facial planes through which this segment passes. A **stratum** of cells is the set of cells of a given power.

If facet f is part of the boundary of cell C and the opposite facet to f is part of the boundary of C', then we may say that C and C' are **connected** by those two facets.

4.4. **Cell Orbits.** As a solid body, every stellation is the union of a set of cells, which in turn is the union of a set of cell orbits. Therefore labels for cell orbits are useful for describing stellations. As with facet orbits, the choice of labels ought to be systematically determined by the constellation.

All the cells in a cell orbit have the same power, which may be used to define the **power** of the cell orbit. A **stratum** of cell orbits is the set of cell orbits of a given power.

For all the cells in one cell orbit, the boundary facets belong to the same set of facet orbits. This set of facet orbits may be termed the **boundary of the cell orbit**. For each such facet orbit, the opposite facet orbit belongs to the boundary of another cell orbit, and we will say that these two cell orbits are **connected** by these two facet orbits. Equivalently, a cell in the first cell orbit is connected, by a facet in the former facet orbit and its opposite facet, to a cell in the second cell orbit.

5. Labeling of Elements of the Constellation

In order to talk about examples of stellations, we need a system for labeling some of the elements of a constellation. The system should not be *ad hoc*; rather, it must be explicit enough to be implemented by a computer. On the other hand, it should not depend on accidental details of the computer software; rather, it should be explicitly defined by the construction of the constellation.

5.1. **Labels for Orbits of Facial Planes.** We begin with the orbits of facial planes of the base polyhedron $\mathcal{P}$. If this is Platonic or convex dual uniform, then there is only one such orbit, which needs no label. If it is convex uniform, then the orbits of facial planes will be labelled by the letters A, B, etc. But in what order? This may be determined by the Wythoff symbol. There are four cases.

- $p|qr$ The orbit of the faces $\{q\}$ is first, followed be that of the faces $\{r\}$; either of these is omitted if the order is equal to 2.
- $pq|r$ The orbit of the faces $\{p\}$ is first, followed by that of the faces $\{q\}$, and then that of the faces $\{2r\}$. either of the first two is omitted if the order is equal to 2.
- $pqr|$ The orbit of faces $\{2p\}$ is first, followed by$\{2q\}$ and then $\{2r\}$.
- $|pqr$ The orbit of faces $\{p\}$is first, followed by $\{q\}$ and then $\{r\}$; any one of these is omitted if the order is equal to 2. Last is the orbit of the snub triangles for which the label is S.

The labels for the opposite orbits are the same, with a raised minus sign as a suffix.

5.2. **Coordinate Axes on a Facial Plane.** In order to display a stellation diagram, one must choose a pair of unit vectors on the facial plane. This choice is also

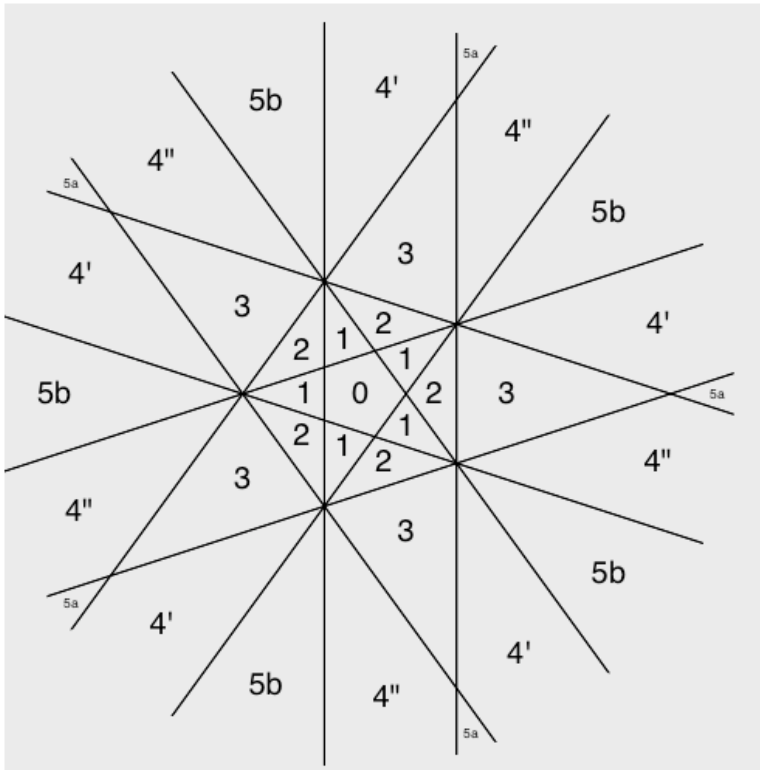

Figure 7. Dodecahedral Face Plane

useful for systematically labeling facet orbits. Here is how we shall choose these vectors, which we shall call $\vec{\xi}$ and $\vec{\eta}$, on a facial plane of one of our polyhedra.

When the polyhedron is specified by an ordinary Wythoff symbol, it may be Platonic or convex uniform, and one or more faces are defined as specified in §2.2. In every case, the center of the face is the point on the plane closest to the center of the polyhedron. The vector $\vec{\xi}$ will be chosen to point from this center to a vertex of the face. (It is immaterial which vertex is chosen.).

When the polyhedron is specified by a dual Wythoff symbol, it may be Platonic or convex dual uniform. There is one kind of positive facial plane, whose center is determined from the Wythoff symbol as in §2.2. In this case, $\vec{\xi}$ will be chosen to point to one vertex of the polyhedron.

The vertices of a convex dual uniform polyhedron are poles of the facial planes of the dual uniform polyhedron (see §2.3). We choose a pole of a facial plane belonging to the first orbit of facial planes, in the ordering given in §5.1.

Finally, $\vec{\eta}$ will be chosen so that $\vec{\xi}$, $\vec{\eta}$, and the unit normal of the oriented plane make up a right-handed triple.

5.3. **Labels for Facet Orbits.** Each facet orbit is associated with an orbit of facial planes, and has been assigned a "power." Each facet orbit in a stratum on a given orbit of facial planes is assigned a **serial number** and a **symmetry class**. To do this, the algorithm makes a list of the facets in a particular stratum, ordered

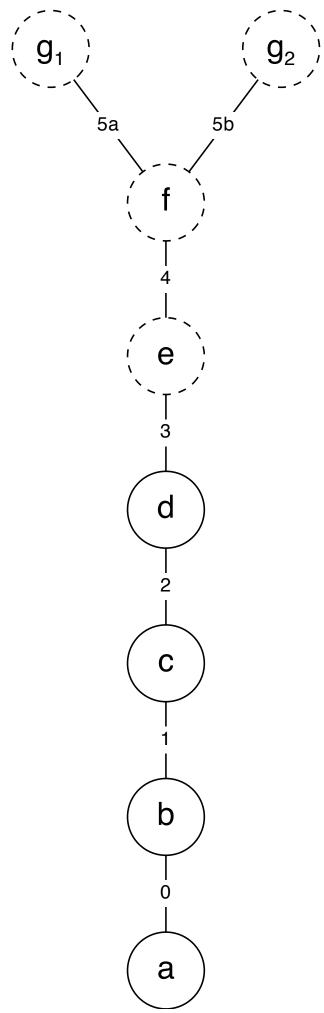

FIGURE 8. Dodecahedral Cell Orbit Graph

as one encounters them starting from the positive $\vec{\xi}$-axis and proceeding in the positive (that is, counterclockwise) direction. The serial numbers start with 1. As the algorithm proceeds through this list, when a facet is encountered, its orbit is given the next serial number, and all the members of the orbit are removed from the list. At the same time, the facet orbit is assigned a symmetry class. If the orbit coincides with its mirror image, then the symmetry class is **middle**. Otherwise, the symmetry class is **left**; the mirror image orbit is given the same serial number and the symmetry class **right**, and its members are also removed from the list of facets.

The label of a facet orbit begins with the numerical power of the orbit. If a label is needed for the orbit of facial planes, it is prepended as a subscript. The power is followed by an alphabetical code (in lower case) for the serial number. If

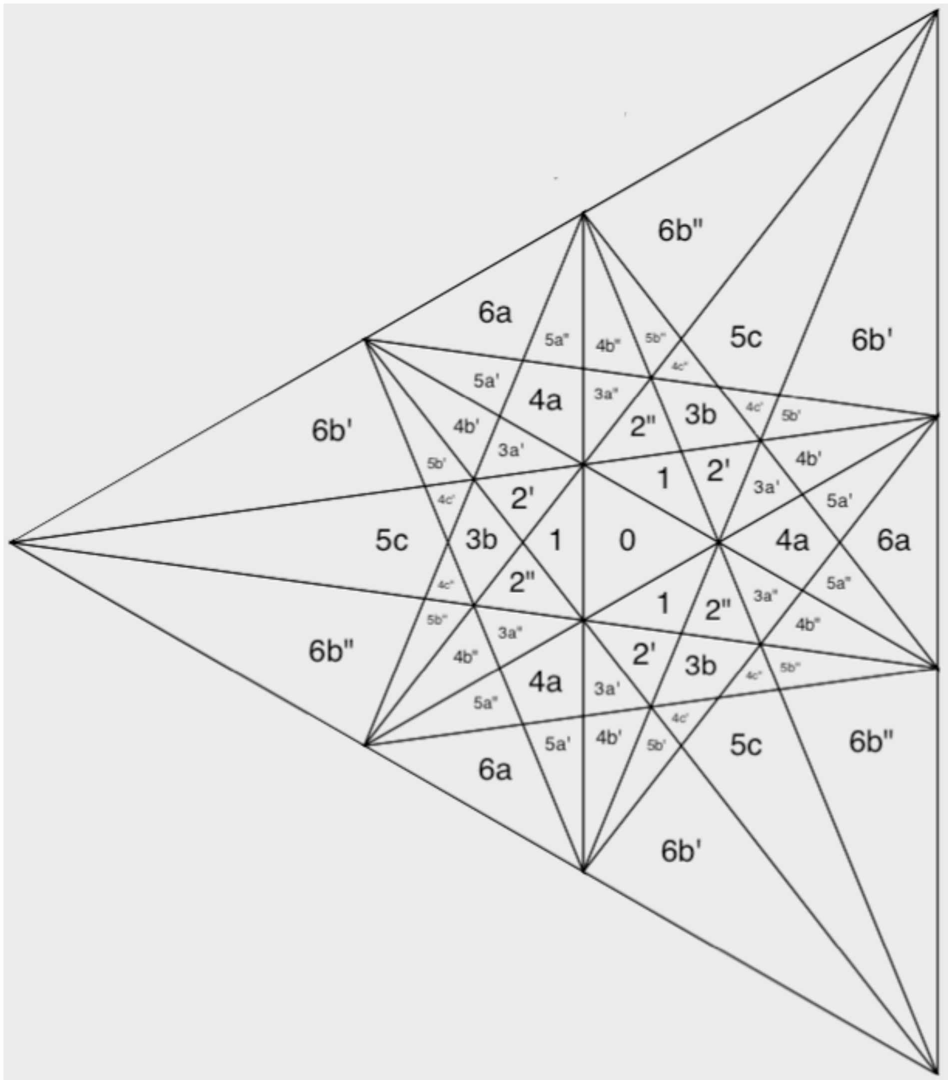

Figure 9. Icosahedral Face Plane, Inner Portion

the symmetry class is "left" then a single prime is appended, and it it is "right" a double prime is appended.

If a stratum contains only one facet orbit, or one orbit and its mirror image, then the code for the serial number may be omitted.

Facet orbits can be ordered lexicographically. The way we shall choose begins with the *sign* of the Facial Plane orbit. This is followed by the letter label of the facial plane orbit, the power, the serial number within the stratum, and finally the symmetry class.

5.4. **Labels For Cell Orbits.** Each cell orbit is characterized by its boundary, that is, the list of facet orbits of the boundary facets of a cell in the cell orbit. This

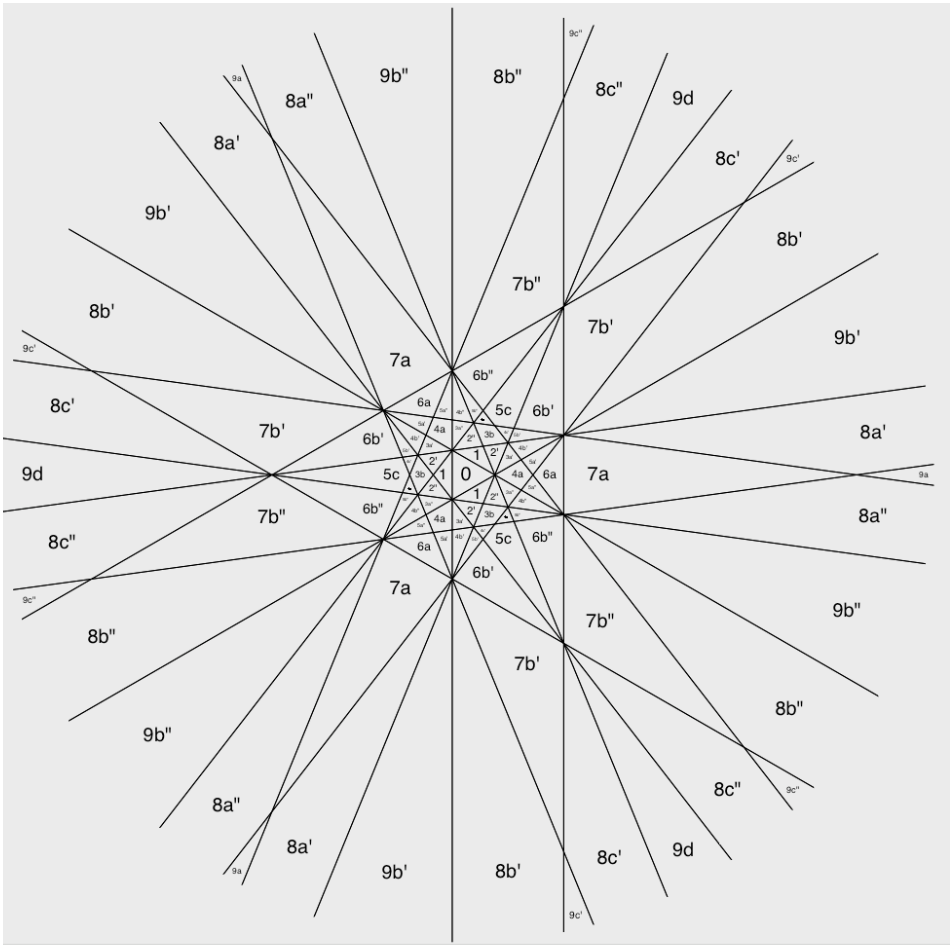

Figure 10. Icosahedral Face Plane in Full

list is assumed to be sorted in the order that was defined in §5.3. Each cell orbit is assigned a serial number within its stratum, and a symmetry class. The cell orbits in a stratum are ordered by the labels of the first facet orbits in their boundary lists. Serial numbers are assigned successively, starting at 1. As with facet orbits, if a cell orbit coincides with its mirror image, then its symmetry class is "middle." Otherwise, the cell orbit and its mirror image get the same serial number; the first one to be found is given the symmetry class "left" and the other one "right."

The label of the cell orbit begins with an alphabetical code for its power, followed by the serial number as a subscript. A single prime is appended if the symmetry class is "left," and a double prime if it is "right." If a stratum contains only one cell orbit, or one cell orbit and its mirror image, then the serial number may be omitted.

6. The Cell Orbit Graph and a Stellation

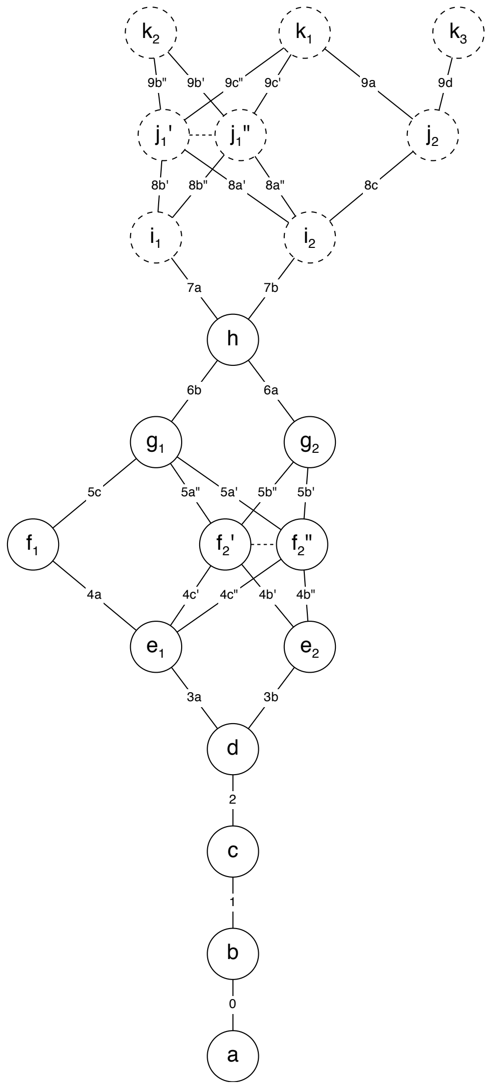

FIGURE 11. Icosahedral Cell Orbit Graph

6.1. **The Cell Orbit Graph.** This is a structure which neatly represents a constellation. It is an undirected graph, with labels for both nodes and arcs. The nodes correspond to cell orbits, and are labeled with the labels of the cell orbits. There is an arc between two nodes if the corresponding cell orbits are connected as defined in §4.4; the arc is labeled with the label of the facet orbit.

6.2. **A Stellation as a Set of Nodes in the Graph.** In §1.2 we stated rules defining a stellation. They can be translated into terms related to the structure of the cell orbit graph, as follows.

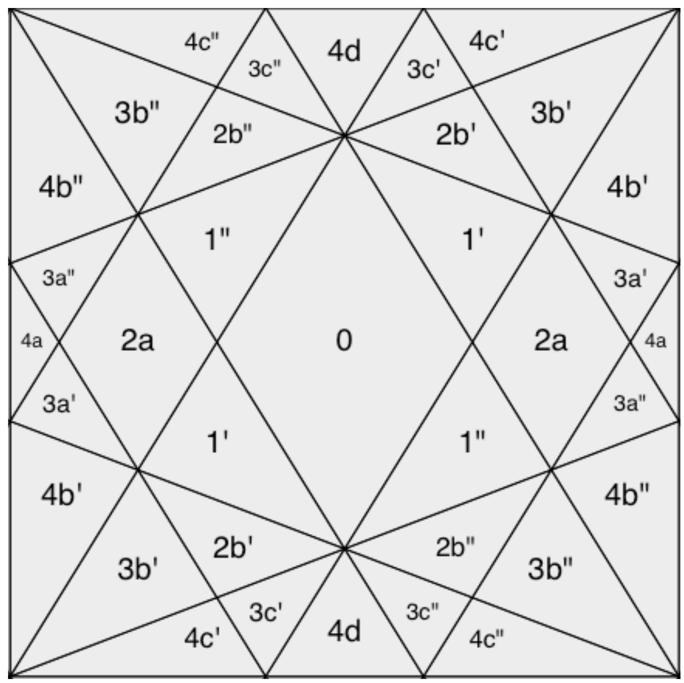

Figure 12. Rhombic Triacontahedron Face Plane: Inner

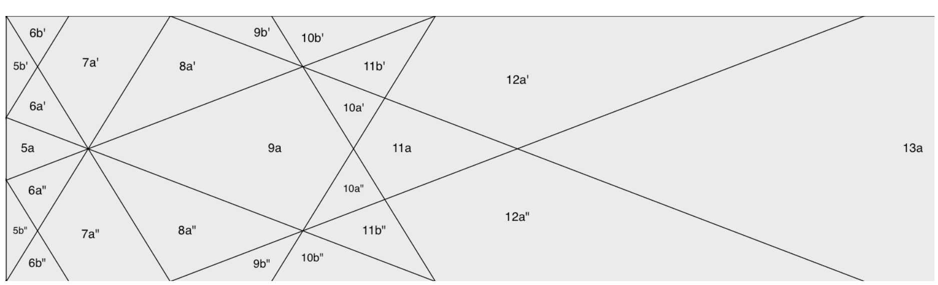

Figure 13. Rhombic Triacontahedron Face Plane: Side Extension

- Rules (1) and (2) are satisfied if and only if the stellation consists of the cells belonging to a set of cell orbits corresponding to the members of a set Σ of nodes in the cell orbit graph.
- Rule (3) is satisfied if and only if, within the complement of Σ, every node is arcwise connected to a non-compact node. If we adjoin a single node,

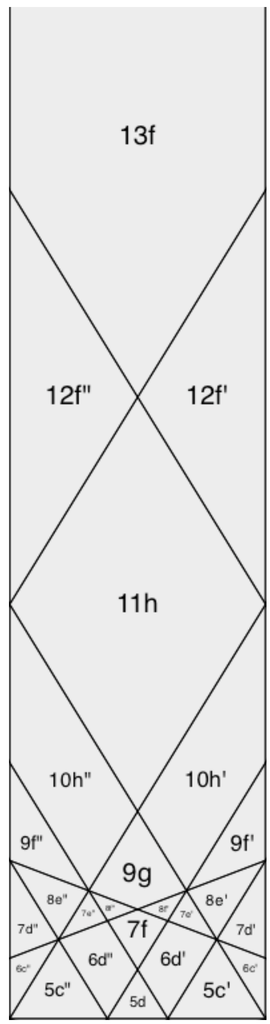

Figure 14. Rhombic Triacontahedron Face Plane: Upper Extensionl

corresponding to the "point at infinity," and connect this to all the non-compact nodes, then we may simplify this criterion: Σ does not contain the node at infinity, and the complement of Σ is arcwise connected.

- Rule (4) is satisfied if and only if Σ is arcwise connected.

6.3. **The Symbol of a Stellation.** The stellation that corresponds to a set Σ of nodes may be designated by a list of the labels of those nodes. If the set contains all the compact nodes up to and including a certain stratum, then the symbol can be abbreviated by capitalizing the letter corresponding to that stratum. to indicate all nodes, compact and noncompact, a superscript "∞" will be attached to the capital letter. When a chiral orbit and its mirror image are both included in the stellation,

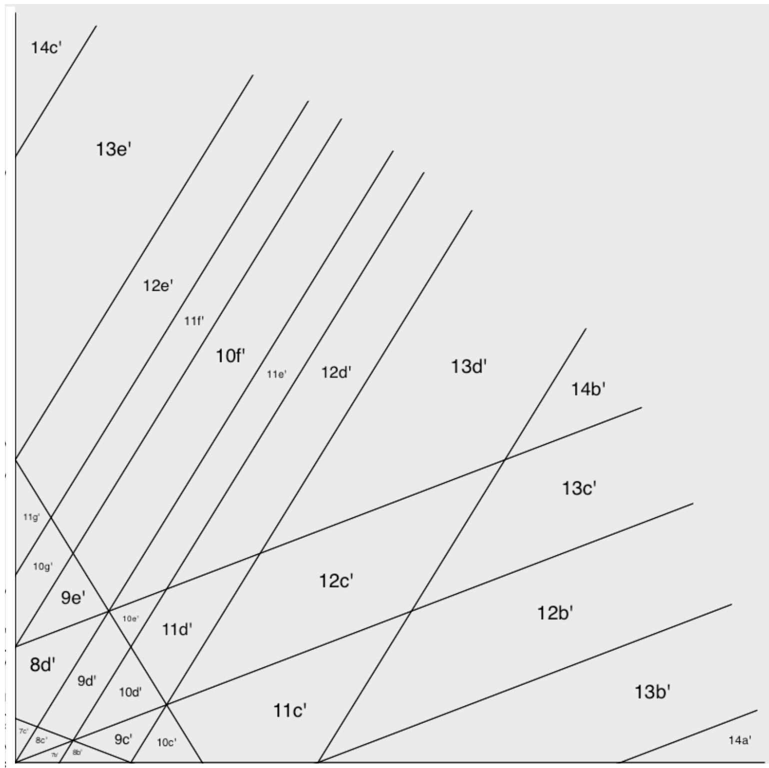

Figure 15. Rhombic Triacontahedron Face Plane: Outer Extension

the two may be represented by a single symbol; we mention an example of this in §7.2.

6.4. **The Symbol and the Facets.** The boundary of the stellation can be determined from Σ, by considering every arc from a member of Σ to a member of the complement. It is labeled by two facet orbits; the one which belongs to the boundary of a cell orbit in Σ contributes to the boundary of the stellation.

6.5. **A Difficulty with Chiral Cell Orbits.** If a base polyhedron has both a special group and a general group of symmetries, its stellations may be defined with respect to either group. The stellations with respect to the general group are necessarily all reversible; those defined with respect to the special group may be reversible or chiral. It may seem that the reversible stellations are the same set of figures whichever group is used. For the icosahedron, this is true; but in general it is not. With respect to the special group, there may be cell orbits which are chiral. The union of such an orbit with its mirror image is a cell orbit with respect to the general group. By Miller's rules, this union is a stellation with respect to the general group, but not with respect to the special group. By firmly established precedent([7]), however, such a union is to be admitted as a stellation. Among other students of stellated polyhedra, the following interpretation of Miller's rules seems to have emerged:

Miller's rule interpreted: A union of cell orbits that is reversible is a stellation if it is a union of orbits with respect to the general group.

There is more discussion of this interpretation in §8.4.

By this interpretation, the cell orbit graph defined with respect to the special group will not in general contain a connected set of nodes for every reversible stellation. We may remedy this by modifying the graph to include **cross-links**; a cross-link is a link between a node representing a chiral cell orbit and is mirror image node. A cross-link is not labeled.

7. Case Studies

Some examples will be useful. The diagrams of Facial Planes do not always fit easily on the page, and so may be broken up into convenient portions.

7.1. The Dodecahedron.

The Facial Plane of the dodecahedron is shown in Figure 7.

The cell orbit graph is shown in Figure 8. In this and other cell orbit graphs, the orbits of non-compact cells are represented by dotted circles. Cross-links are shown by dotted lines.

It is easy to see that there are just 4 compact stellations of the dodecahedron:

- A**:** the dodecahedron itself;
- B**:** the small stellated dodecahedron;
- C**:** the great dodecahedron;
- D**:** the great stellated dodecahedron.

One may also discern that 4 non-compact stellations exist.

7.2. The Icosahedron.

The Facial Plane of the icosahedron is shown in two figures. Figure 9 shows the part of the facial plane inside the triangular faces of a Great Icosahedron. The full diagram is shown, at a reduced scale, in Figure 10.

The correspondence between the facet orbit labels of this paper and those of [8] is given in Appendix A.

The cell orbit graph is shown in Figure 11. The correspondence between the cell orbit labels of this paper and those of [8] is given in Appendix A.

It is well known([8]) that there are 32 centrally symmetric stellations of the icosahedron and 27 pairs of chiral stellations. We mention only a few:

- A**:** the icosahedron itself;
- G**:** the great icosahedron;
- Ef_2'**:** the compound of five tetrahedra;
- Ef_2**:** the compound of ten tetrahedra.

Note how, in the last example, f_2 stands for $\mathsf{f}_2'\mathsf{f}_2''$.

7.3. The Rhombic Triacontahedron.

The rhombic triacontahedron (or "R30") is a Catalan polyhedron, $\mathsf{D}(2|3\ 5)$. Among its stellations, perhaps the most elegant is the compound polyhedron of five cubes inscribed in a dodecahedron. Thus the facial plane of the R30 includes a face of one of those cubes; it is shown in Figure 12. If two sides of the cubic face are extended to the right, they enclose another part of the diagram, shown in Figure 13. If the other two sides of the cubic face are extended upwards, they enclose a part of the diagram shown in Figure 14. Finally, between the latter two parts of the diagram is a corner shown in Figure 15.

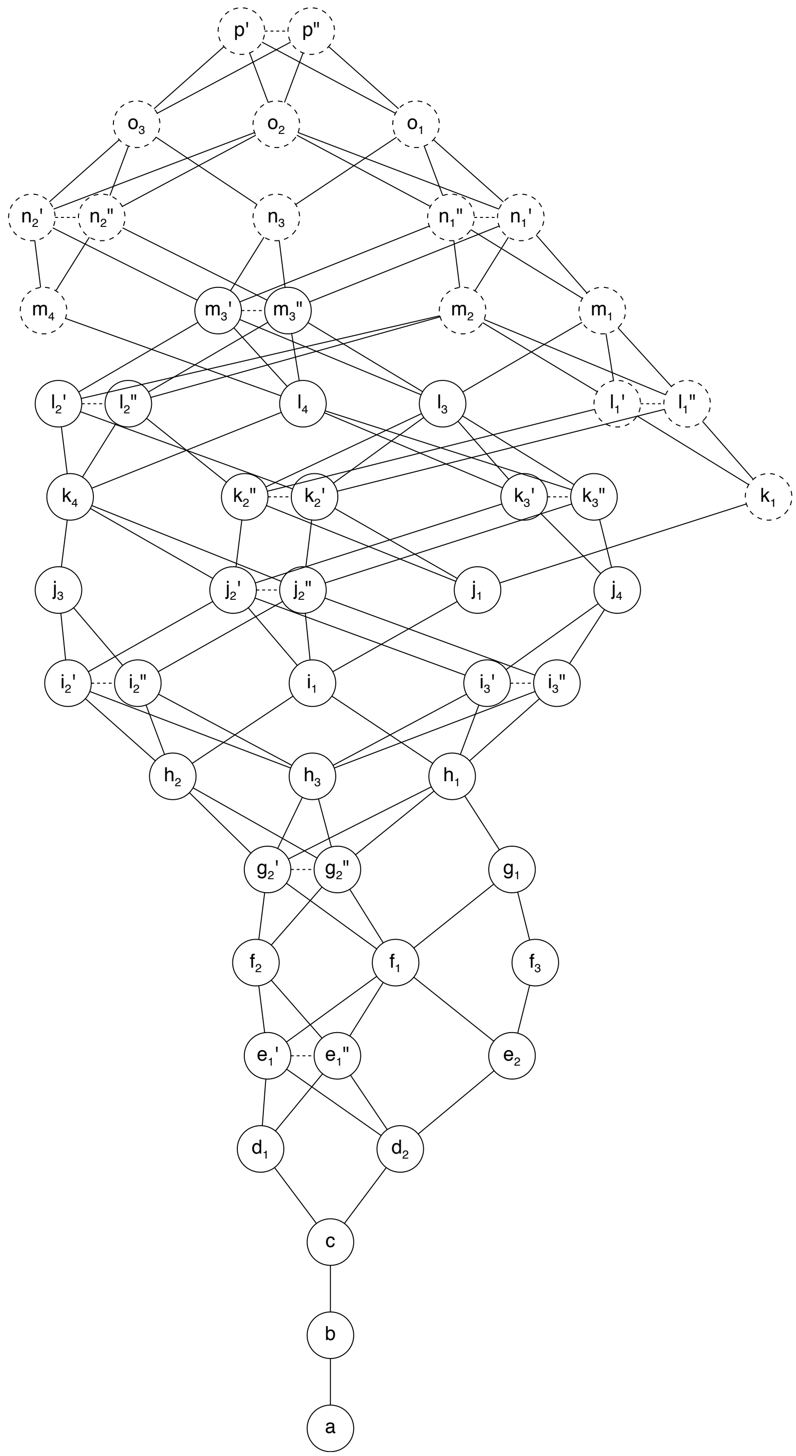

Figure 16. Rhombic Triacontahedron Cell Orbit Graph

A diagram of the cell orbit graph is given in Figure 16. Because this diagram is crowded with arcs representing facet orbits, I have not labelled those arcs. Instead, the tables in Appendix B specify the arcs which join each connected pair of nodes.

Many interesting polyhedra are stellations of R30. In our notation, the compound of 5 cubes inscribed in a dodecahedron is **E**. The small and great stellated R30 are, respectively, $\mathbf{Cd_2e_2f_3}$ ([25], Photo 27) and $\mathbf{Gh_2h_3i_2j_3}$ ([25], Photo 32). Wenninger has also constructed models of several noncompact stellations (trucated, of course); Table 3 gives their symbols.

Illustration in [25]	symbol
Photo 61	$\mathbf{Gh_1h_2i_1j_1k_1}$
Photo 62	$\mathbf{Ij_2j_3j_4k_3k_4}$
Photo 63	$\mathbf{Kl_1l_2l_3m_1m_2}$
Photo 65	$\mathbf{M^\infty n_2}$
Photo 66	$\mathbf{N^\infty}$

Table 3. Noncompact Stellations of R30

8. Counting Stellations

The stellations of a polyhedron correspond, as we saw in §6.2, to sets of nodes in the cell orbit graph $\mathcal{G}$. Specifically, a stellation corresponds to a connected set of nodes whose complement is also connected and includes nodes corresponding to orbits of unbounded cells.

Coxeter *et al.*[7, 8] enumerated the minimal cut sets (corresponding to compact stellations) of the cell orbit graph of the icosahedron in [8], by *ad hoc* methods. For a cell orbit graph like Figure 16, one wants to obtain the number of minimal cut sets without having to write them all down, even in a computer file.

8.1. **Developing a Strategy.** Similar problems have been solved by constructing **binary decision diagrams**, as expounded by Knuth [16]. A binary decision diagram (BDD) is a directed acyclic graph with two "sink nodes" labeled $\top$ and $\bot$; every other node has two arcs leading out of it, which we label LO and HI. Each nonsink node is labeled with a non-negative integer, and when an arc goes from a node labeled i to a node labeled j, we require that $i < j$. The graph is connected, and there is a single "root" node whose label is 0. Knuth also requires that the graph be "reduced," which means that for any non-sink node, LO and HI must not lead to the same destination, and and there cannot be two distinct nodes with the same label and the same destinations of LO and HI.

One application of binary decision diagrams is to enumerate certain classes of subsets of a combinatorial structure. In this context, a path from the root node to $\top$ corresponds to a subset: an arc HI from a node labeled i signifies that element i is included in the subset, and an arc LO from that node signifies that i is excluded. In [16], §7.1.4, in the section "Boolean programming and beyond," Figure 22 contains a small graph and a BDD for counting its connected subgraphs. Exercise 55 asks for an algorithm for constructing such a BDD. The answer as given suggests some useful methods for solving such problems. Of course, there is a great variety of possible problems, so the details will be different.

- In "exercise 55," the elements to be included or excluded are arcs of a graph. In our problem, the elements are nodes of the cell orbit graph.
- At each stage of building the BDD, there is a distinction between the nodes that have been considered and those that have not. Among the nodes that have been considered, there is an important subset, which Knuth calls the "frontier." The definition of this frontier is different for different problems.
- In our enumeration problem, the size of the BDD can be very large. We prefer to have only a small part of it exist in memory at any time, namely that which pertains to two successive subsets of nodes of the cell orbit graph.
- Knuth suggests that the paths to $\top$ be counted by building the entire BDD and working back. Because we wish to maintain only a small part of the BDD in our program data, we need to count paths by working forward. We will prune nodes that we know cannot reach $\top$.
- Knuth suggests that in the unreduced BDD, nodes correspond to "partitions of [a frontier] that correspond to connectedness relations that have arisen because of previous branches." This statement is a bit loose, and needs to be firmed up for the given problem.

We begin the process of "firming up" by noting that the nodes in $\mathcal{G}$ are ordered by "power." We say that node x is **below** node y, and likewise y is **above** x, if x and y are connected by an arc and x has lower power than y. We say that a subset $\mathcal{S}$ of $\mathcal{G}$ is **well-founded** if for any $x \in \mathcal{S}$ and any y which is below x, we also have $y \in \mathcal{S}$. The subsets of $\mathcal{G}$ which we will use in the stages of building a BDD will always be well-founded.

The **frontier** of a well-founded $\mathcal{S}$ is the set of nodes in $\mathcal{S}$ which have arcs connecting them to nodes in $\mathcal{G} \setminus \mathcal{S}$. Note that if $x \in \mathcal{S}$, $y \in \mathcal{G} \setminus \mathcal{S}$, and there is an arc from x to y, then y must be above x. The frontier of $\mathcal{S}$ will be denoted $\partial\mathcal{S}$.

We will define the BDD using a sequence of well-founded subsets of the nodes of $\mathcal{G}$; each subset will include one more node than its predecessor. For each well-founded subset $\mathcal{S}$, there will be a set of BDD nodes, each of which denotes a collection of subsets of $\mathcal{S}$. In some way, to be "firmed up" presently, this collection is to be characterized in terms of $\partial\mathcal{S}$.

The next well-founded subset after $\mathcal{S}$ is constructed by choosing a node $x \in \mathcal{G} \setminus \mathcal{S}$, subject to the condition that $\mathcal{S} \cup \{x\}$ is also well-founded. For each BDD node associated with $\mathcal{S}$, the HI and LO branches stand for the situations in which x respectively is, or is not, added to the subsets denoted by that BDD node.

Our next step in firming up the algorithm must be to decide what subsets of a well-founded $\mathcal{S}$ are of interest, and how they should be aggregated in collections corresponding to BDD nodes. We ask ourselves what subsets of a well-founded $\mathcal{S}$ might give rise to stellations.

Definition 8.1. Let $\mathcal{S}$ be a well-founded subset of $\mathcal{G}$. A subset Q of $\mathcal{S}$ is a **protostellation** for $\mathcal{S}$ if

(1) every connected component of Q intersects $\partial\mathcal{S}$;
(2) every connected component of $\mathcal{S} \setminus Q$ intersects $\mathcal{G}_u \cup \partial\mathcal{S}$.

Proposition 8.2. *A protostellation belongs to* $\mathrm{Stell}(\mathcal{G})$ *if and only if it is a connected subgraph of* $\mathcal{G}$*.*

Proof. The condition is necessary, because every stellation is connected. Conversely, let Q be a protostellation for $\mathcal{S}$. Then every component of $\mathcal{S} \setminus Q$ either contains a node of $\mathcal{G}_u$ or else contains a node x of $\partial\mathcal{G}$. In the latter case, if x is not in $\mathcal{G}_u$ then it is connected to a node in the next higher stratum; and so on, until a node of $\mathcal{G}_u$ is reached. Thus every component of $\mathcal{G} \setminus Q$ contains a node of $\mathcal{G}_u$. If Q is connected as a subgraph of $\mathcal{G}$, then it satisfies all the conditions for a member of $\mathrm{Stell}(\mathcal{G})$. $\square$

Proposition 8.3. *Let $\mathcal{S}$ be a well-founded subset of $\mathcal{G}$. If $\Sigma \in \mathrm{Stell}(\mathcal{G})$ and $\Sigma \cap \partial\mathcal{S} \neq \emptyset$, then $\Sigma \cap \mathcal{S}$ is a protostellation for $\mathcal{S}$.*

Proof. First, let Σ' be a connected component of $\Sigma \cap \mathcal{S}$. If Σ' did not intersect $\partial\mathcal{S}$, then it would have to be a connected component of Σ and also a proper subset of Σ, contrary to the condition that Σ must be connected.

Second, consider $T = \mathcal{G} \setminus \Sigma$; it is connected, and contains ω. Every element x of T must be on a path in T which reaches ω. If $x \in \mathcal{S}$ then this path needs to pass through $\partial\mathcal{S} \cup \mathcal{G}_u$. Therefore $T \cap \mathcal{S}$ must satisfy condition (2) of Definition 8.1. $\square$

In terms of subsets of $\mathcal{S}$, our strategy is to find protostellations for $\mathcal{S}$. If a protostellation is connected, then count it as a member of $\mathrm{Stell}(\mathcal{G})$. Whether it is connected or not, it may serve as part of a protostellation for $\mathcal{S} \cup x$.

To continue with the formulation of our strategy, we want to define a way of collecting sets of protostellations for $\mathcal{S}$, based on their intersections with $\partial\mathcal{S}$. A protostellation Q for $\mathcal{S}$ determines a subset of $\partial\mathcal{S}$, namely $L = Q \cap \partial\mathcal{S}$; moreover, it determines a partition of L into subsets, namely the intersections of $\partial\mathcal{S}$ with the connected components of Q. Likewise, $\partial\mathcal{S} \setminus L$ is partitioned into subsets, namely the non-empty intersections with connected components of $\mathcal{S} \setminus Q$. These subsets of $\partial\mathcal{S}$ must be distinguished, depending on whether the connected components contain nodes of $\mathcal{G}_u$ or not. Thus, partitions of $\partial\mathcal{S}$ into subsets, grouped into three kinds, are a means for grouping protostellations. We formulate this "grouping of subsets" as follows.

Definition 8.4. Let $\mathcal{S}$ be a well-founded subset of $\mathcal{G}$. Then a **distribution within** $\partial\mathcal{S}$ is an ordered set of three elements:

$$\mathbf{D} = (D_L, D_M, D_N) \tag{8.1}$$

where

$$\begin{aligned} D_L &= \{L_1, \ldots, L_a\}, \\ D_M &= \{M_1, \ldots, M_b\}, \\ D_N &= \{N_1, \ldots, N_c\}, \end{aligned} \tag{8.2}$$

such that

$$\partial\mathcal{S} = L_1 \cup \cdots \cup L_a \cup M_1 \cup \cdots \cup M_b \cup N_1 \cup \cdots \cup N_c$$

is a partition of $\partial\mathcal{S}$ into disjoint non-empty sets. The subsets L_i, M_i, and N_i will be called **parcels**. The sets D_L, D_M, and D_N will be called the **sections** of $\mathbf{D}$. Although (to repeat) the parcels are non-empty, we allow a section to be empty.

In the terminology of MacMahon ([17], Chapter II), what we are considering is a "distribution of objects of type (1^n) into parcels of type (abc)." The "objects of type (1^n)" are the nodes in $\partial\mathcal{S}$. The "type" merely specifies that the nodes are distinguishable from one another. The parcel type signifies that the order of the parcels within a section is immaterial but the sections are ordered.

8.2. **Distributions and protostellations.** Let us make explicit the relation between these "distributions" and protostellations. In what follows, $\mathcal{S}$ be a well-founded subset of $\mathcal{G}$.

Proposition 8.5. *Let Q be a protostellation for $\mathcal{S}$. Define L_i, M_i, and N_i as follows:*

(1) *$L_1, \ldots, L_a$ are the intersections of $\partial\mathcal{S}$ with the connected components of Q;*
(2) *$M_1, \ldots, M_b$ are the intersections of $\partial\mathcal{S}$ with those components of $\mathcal{S}\setminus Q$ which do not intersect $\mathcal{G}_u$; and*
(3) *$N_1, \ldots N_c$ are the non-empty intersections of $\partial\mathcal{S}$ with those components of $\mathcal{S} \setminus Q$ which do intersect $\mathcal{G}_u$.*

Define D_L, D_M, and D_N as in (8.2), and define $\mathbf{D}$ as in (8.1). Then $\mathbf{D}$ is a distribution within $\partial\mathcal{S}$.

Proof. Part (1) of Definition 8.1 implies that the sets L_i are nonempty. Likewise, part (2) implies that the sets M_i are nonempty. It is clear that the "parcels" L_i, M_i, and N_i make up a partition of $\mathcal{G}_p$ into disjoint nonempty sets, and so the conditions of Definition 8.4 are satisfied. □

Definition 8.6. Let Q be a protostellation for $\mathcal{S}$. Then the **distribution of** Q, denoted $\mathrm{Dist}(Q)$, is the distribution $\mathbf{D}$ of Proposition 8.5.

Definition 8.7. For any distribution $\mathbf{D}$ within $\partial\mathcal{S}$, we define[8] $\mathrm{Nebula}(\mathcal{G}, \mathbf{D})$ as the set of protostellations Q for $\mathcal{S}$ such that $\mathrm{Dist}(Q) = \mathbf{D}$. We also define $N(\mathcal{G}, \mathbf{D})$ as the number of elements of $\mathrm{Nebula}(\mathcal{G}, \mathbf{D})$.

Proposition 8.8. *If the distribution $\mathbf{D}$ satisfies $a = |D_L| = 1$, then $\mathrm{Nebula}(\mathcal{G}, \mathbf{D})$ is a subset of $\mathrm{Stell}(\mathcal{G})$. On the other hand, if $a \neq 1$, then $\mathrm{Nebula}(\mathcal{G}, \mathbf{D})$ is disjoint from $\mathrm{Stell}(\mathcal{G})$.*

Proof. The condition that $a = 1$ simply says that for every element Q of $\mathrm{Nebula}(\mathcal{G}, \mathbf{D})$ the set Q is connected. By Proposition 8.2, this implies that $Q \in \mathrm{Stell}(\mathcal{G})$. On the other hand, if $a = 0$ then Q must be empty, which disqualifies it as a stellation, and if $a > 1$ then Q must not be connected. □

Proposition 8.9. *Let Q be a protostellation for $\mathcal{S}'$ and let $Q_0 = Q \cap \mathcal{S}$. Then Q_0 is a protostellation for $\mathcal{S}$.*

Proof. Let C_0 be a connected component of Q_0. Then C_0 is contained in a connected component C of Q. By hypothesis some node of $\mathcal{G} \setminus \mathcal{S}'$ is connected to C. If this connection is to x, then either C_0 is empty or C_0 is connected to x and thus to $\mathcal{G} \setminus \mathcal{S}$. If the connection of $\mathcal{G} \setminus \mathcal{S}'$ is to some other node of C, then this is also a node of C_0, which is thus connected to $\mathcal{G} \setminus \mathcal{S}$.

Similarly, let C_0 be a connected component of $\mathcal{S} \setminus Q_0$ containing no nodes of $\mathcal{G}_u$, and let C be the connected component of $\mathcal{S}' \setminus Q$ containing C. By hypothesis either some node of $\mathcal{G} \setminus \mathcal{S}'$ is connected to C, or C contains a node of $\mathcal{G}_u$. If some node of $\mathcal{G} \setminus \mathcal{S}'$ is connected to C, then as in the previous paragraph, this implies a connection of C_0 to $\mathcal{G} \setminus \mathcal{S}$. If C contains a node of $\mathcal{G}_u$, then this node must be x, and as before it implies that C_0 is connected to $\mathcal{G} \setminus \mathcal{S}$. □

[8] Why "nebula"? Because such a set Q may or may not coalesce into a star, that is, a stellation.

Here	As in [8]
0	**0**
1	**1**
2’, 2”	**2**
3a’, 3a”	**4**
3b	**3**
4a	**7**
4b’	*5*
4b”	5
4c’	6
4c”	*6*
5a’	9
5a”	*9*
5b’	10
5b”	*10*
5c	**8**
6a	**12**
6b’,6b”	**11**
7a	**13**
7b’, 7b”	**13**

Table 4. Facet Orbits of the Icosahedron

Proposition 8.8 shows that $\mathrm{Stell}(\mathcal{G})$ is the union of the sets $\mathrm{Nebula}(\mathcal{G}, \mathbf{D})$ for certain distributions $\mathbf{D}$. We must now consider the relation between distributions within the boundaries of two successive well-founded sets in our sequence; let these be $\mathcal{S}$ and $\mathcal{S}' = \mathcal{S} \cup \{x\}$. We will presuppose these definitions in the rest of this subsection.

We need to define criteria $\mathcal{C}_{LO}$ and $\mathcal{C}_{HI}$ for a distribution $\mathbf{D}$ within $\partial\mathcal{S}$, with the following properties.

- If $\mathcal{C}_{LO}$ is satisfied, then for every protostellation $Q \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D})$ we also have that Q is a protostellation for $\mathcal{S}'$; whereas if $\mathcal{C}_{LO}$ is not satisfied, then no protostellation $Q \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D})$ is a protostellation for $\mathcal{S}'$.
- Similarly, if $\mathcal{C}_{HI}$ is satisfied, then for every protostellation $Q \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D})$ we also have that $Q \cup \{x\}$ is a protostellation for $\mathcal{S}'$; whereas if $\mathcal{C}_{HI}$ is not satisfied, then for no protostellation $Q \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D})$ is $Q \cup \{x\}$ a protostellation for $\mathcal{S}'$.

We also need to define distributions $\mathrm{Prop}_{LO}(\mathbf{D}, x)$ and $\mathrm{Prop}_{HI}(\mathbf{D}, x)$, both within $\partial\mathcal{S}'$, with the following properties.

- If $\mathcal{C}_{LO}$ is satisfied, then $\mathbf{D}' = \mathrm{Prop}_{LO}(\mathbf{D}, x)$ is defined as a distribution within $\partial\mathcal{S}'$ and $\mathrm{Nebula}(\mathcal{G}, \mathbf{D}') \supseteq \mathrm{Nebula}(\mathcal{G}, \mathbf{D})$.
- If $\mathcal{C}_{HI}$ is satisfied, then $\mathbf{D}' = \mathrm{Prop}_{HI}(\mathbf{D}, x)$ is defined as a distribution within $\partial\mathcal{S}'$ and $\mathrm{Nebula}(\mathcal{G}, \mathbf{D}') \supseteq \{Q \cup \{x\} | Q \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D})\}$.

With these criteria and partial functions, the definition of the BDD can be completed. We need to specify the targets of the HI and LO branches from the BDD node corresponding to the distribution $\mathbf{D}$ for $\mathcal{S}$.

Here	As in [8]
a	**A**
b	**b**
c	**c**
d	**d**
e_1	$\mathbf{e_2}$
e_2	$\mathbf{e_1}$
f_1	$\mathbf{f_2}$
f_2'	f_1
f_2''	f_1
g_1	$\mathbf{g_2}$
g_2	$\mathbf{g_1}$
h	**h**

Table 5. Cell Orbits of the Icosahedron

- If $\mathcal{C}_{LO}$ is satisfied by $\mathbf{D}$, then the LO branch from this node goes to the node for the distribution $\mathbf{D'} = \mathrm{Prop}_{LO}(\mathbf{D}, x)$. Otherwise it goes to $\perp$.
- If $\mathcal{C}_{HI}$ is satisfied by $\mathbf{D}$, then the HI branch from this node goes to the node for the distribution $\mathbf{D'} = \mathrm{Prop}_{HI}(\mathbf{D}, x)$. Otherwise it goes to $\perp$.

Before giving the details of $\mathcal{C}_L O$, $\mathcal{C}_H I$, Prop_{LO}, and Prop_{HI}, let us consider the relation between $\partial\mathcal{S}$ and $\partial\mathcal{S}'$, given that $\mathcal{S}' = \mathcal{S} \cup \{x\}$. We say that a node y of $\mathcal{G}$ is **shadowed in** $\mathcal{S}$ **by** x, if $y \in \partial\mathcal{S}$ and the only arc from y to a node not in $\mathcal{S}$ is to x. With this definition, we can say that $\partial\mathcal{S}'$ is formed from $\partial\mathcal{S}$ by including x and excluding the nodes that are shadowed in $\mathcal{S}$ by x. In a similar way, if S is a subset of $\partial\mathcal{S}$, we say that S is **shadowed in** $\mathcal{S}$ **by** x if all the nodes of S are shadowed in $\mathcal{S}$ by x.

Definition 8.10. Let $\mathbf{D}$ be a distribution within $\partial\mathcal{S}$. We let $\mathcal{C}_{LO}(\mathcal{S}, \mathbf{D}, x)$ be true if none of the parcels of D_L are shadowed in $\mathcal{S}$ by x, and false otherwise. Similarly, we let $\mathcal{C}_{HI}(\mathcal{S}, \mathbf{D}, x)$ be true if none of the parcels of D_M are shadowed in $\mathcal{S}$ by x, and false otherwise.

Proposition 8.11. *Let* $\mathbf{D}$ *be a distribution within* $\partial\mathcal{S}$*, and let* Q *be a protostellation in* $\mathrm{Nebula}(\mathcal{G}, \mathbf{D})$*. Then*

(a) Q *is a protostellation for* $\mathcal{S}'$ *if and only if* $\mathcal{C}_{LO}(\mathcal{S}, \mathbf{D}, x)$ *is true.*
(b) $Q \cup \{x\}$ *is a protostellation for* $\mathcal{S}'$ *if and only if* $\mathcal{C}_{HI}(\mathcal{S}, \mathbf{D}, x)$ *is true.*

Proof. (a) Suppose $\mathcal{C}_{LO}(\mathcal{S}, \mathbf{D}, x)$ is true. To show that Q is a protostellation for $\mathcal{S}'$, we must consider the connected components of Q and those of $\mathcal{S}' \setminus Q$. By hypothesis, no parcel in D_L is shadowed in $\mathcal{S}$ by x. This is as much as to say that every connected component of Q intersects $\partial\mathcal{S}'$. Now consider a connected component C' of $\mathcal{S}' \setminus Q$.

- If $x \in C'$, then clearly C' intersects $\partial\mathcal{S}'$.
- If $x \notin C'$, then C' is a connected component of $\mathcal{S} \setminus Q$. By hypothesis, either C' contains a node belonging to $\mathcal{G}_u$ or C' intersects $\partial\mathcal{S}$ in a set P that is a parcel in D_M. P is not connected to x, so is not shadowed by x, and so is a part of $\partial\mathcal{S}'$.

In either case, C' has a non-empty intersection with $\mathcal{S}'$. It follows that Q is a protostellation for $\mathcal{S}'$.

On the other hand, if $\mathcal{C}_{LO}(\mathcal{S}, \mathbf{D}, x)$ is false, then some parcel in D_L is shadowed in $\mathcal{S}$ by x. This parcel is the intersection with $\partial\mathcal{S}$ of some connected component C of Q. It follows that $C \cap \partial\mathcal{S}' = \emptyset$. Thus Q is not a protostellation for $\mathcal{S}'$. This completes the proof of (a).

(b) The proof of (b) is very similar in structure to that of (a), but the differences in detail cannot be conveyed with a wave of the hand.

Suppose $\mathcal{C}_{HI}(\mathcal{S}, \mathbf{D}, x)$ is true. To show that $Q \cup \{x\}$ is a protostellation for $\mathcal{S}'$, we must consider the connected components of $Q \cup \{x\}$ and those of $\mathcal{S}' \setminus (Q \cup \{x\})$; note that the latter set is the same as $\mathcal{S} \setminus Q$. We consider the components of the latter set first. By hypothesis, no parcel in D_M is shadowed in $\mathcal{S}$ by x. This is as much as to say that every connected component of $\mathcal{S} \setminus Q$ that does not intersect $\mathcal{G}_u$ intersects $\partial\mathcal{S}'$. Now consider a connected component C' of $Q \cup \{x\}$.

- If $x \in C'$, then clearly C' intersects $\partial\mathcal{S}'$.
- If $x \notin C'$, then C' is a connected component of Q. By hypothesis, C' intersects $\partial\mathcal{S}$ in a set P that is a parcel in D_L. P is not connected to x, so is not shadowed by x, and so is a part of $\partial\mathcal{S}'$.

In either case, C' has a non-empty intersection with $\mathcal{S}'$. It follows that Q is a protostellation for $\mathcal{S}'$.

On the other hand, if $\mathcal{C}_{HI}(\mathcal{S}, \mathbf{D}, x)$ is false, then some parcel in D_M is shadowed in $\mathcal{S}$ by x. This parcel is the intersection with $\partial\mathcal{S}$ of some connected component C of $\mathcal{S} \setminus Q$, such that C does not intersect $\mathcal{G}_u$. It follows that $C \cap \partial\mathcal{S}' = \emptyset$. Thus Q is not a protostellation for $\mathcal{S}'$. This completes the proof of (b).

□

It is time to define the propagation functions. Let $\mathbf{D} = (D_L, D_M, D_N)$ be a distribution within $\partial\mathcal{S}$.

- If $\mathcal{C}_{LO}(\mathcal{S}, \mathbf{D}, x)$ is true, then $\mathrm{Prop}_{LO}(\mathbf{D}, x)$ is the triple $\mathbf{D}' = (D'_L, D'_M, D'_N)$ where
  - **L:** D'_L contains, for every parcel $P \in D_L$, the parcel $P' = P \cap \partial\mathcal{S}'$;
  - **M:** if x is a bounded cell orbit and no parcel in D_N is connected to x, then D'_M contains a parcel which is the intersection of $\partial\mathcal{S}'$ with the union of $\{x\}$ and all parcels in D_M that are connected to x; moreover, for every parcel P in D_M that is not connected to x, D'_M contains the parcel $P' = P \cap \partial\mathcal{S}'$;
  - **N:** if x is unbounded or some parcel in D_N is connected to x, then D'_N contains a parcel which is the union of $\{x\}$ and all parcels in $D_M \cup D_N$ that are connected to x; D'_N also includes every parcel P in D_N that is not connected to x;

  no other parcels are included except those specified above.
- If $\mathcal{C}_{HI}(\mathcal{S}, \mathbf{D}, x)$ is true, then $\mathrm{Prop}_{HI}(\mathbf{D}, x)$ is the triple $\mathbf{D}' = (D'_L, D'_M, D'_N)$ where
  - **L:** D'_L contains a parcel which is the intersection of $\partial\mathcal{S}'$ with the union of $\{x\}$ and all parcels in D_L that are connected to x; moreover, for every parcel P in D_L that is not connected to x, D'_L contains the parcel $P' = P \cap \partial\mathcal{S}'$;
  - **M:** D'_M contains, for every parcel $P \in D_M$, the parcel $P' = P \cap \partial\mathcal{S}'$;

N: for every parcel $P \in D_N$ such that $P' = P \cap \partial\mathcal{S}'$ is not empty, D'_N includes P';

no other parcels are included except those specified above.

The next two propositions are immediate consequences of the definitions.

Proposition 8.12. *Let* $\mathbf{D}$ *be a distribution within* $\partial\mathcal{S}$*, and let* Q *be a protostellation in* $\mathrm{Nebula}(\mathcal{G}, \mathbf{D}$*. Then*

(a) *If* $\mathcal{C}_{LO}(\mathcal{S}, \mathbf{D}, x)$ *is true, and* $\mathbf{D}' = \mathrm{Prop}_{LO}(\mathbf{D}, x)$*, then* $Q \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D}')$*.*
(b) *If* $\mathcal{C}_{HI}(\mathcal{S}, \mathbf{D}, x)$ *is true, and* $\mathbf{D}' = \mathrm{Prop}_{HI}(\mathbf{D}, x)$*, then* $Q \cup \{x\} \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D}')$*.*

Proposition 8.13. *Let* $\mathbf{D}'$ *be a distribution within* $\partial\mathcal{S}'$*, and let* $Q' \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D}')$*.*

(a) *If* $x \notin Q'$*, let* $\mathbf{D}$ *be the distribution within* $\mathcal{S}$ *such that* $Q' \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D})$*. Then* $\mathcal{C}_{LO}(\mathcal{S}, \mathbf{D}, x)$ *is true, and* $\mathrm{Prop}_{LO}(\mathbf{D}, x) = \mathbf{D}'$*.*
(b) *If* $x \in Q'$*, let* $Q = Q' \setminus \{x\}$*; let* $\mathbf{D}$ *be the distribution within* $\mathcal{S}$ *such that* $Q \in \mathrm{Nebula}(\mathcal{G}, \mathbf{D})$*. Then* $\mathcal{C}_{HI}(\mathcal{S}, \mathbf{D}, x)$ *is true, and* $\mathrm{Prop}_{HI}(\mathbf{D}, x) = \mathbf{D}'$*.*

8.3. **Notes on the Implementation.**

8.3.1. *Data Types.* Distributions are the most complicated data type involved in the counting algorithm. The parcels of a distribution within $\mathcal{S}$ are a *set partition* of $\partial\mathcal{S}$. The elements of $\partial\mathcal{S}$ are referred to by integers $1, ..., n$; there is a data structure for a "Frontier" that includes, among other things, information on the orbits indicated by these numbers. According to Knuth([15],§7.2.1.5), a set partition is conveniently encoded a a *restricted growth string*. This is a sequence of integers $a_1 a_2 \ldots a_n$, where n is the size of $\partial\mathcal{S}$. The integers are chosen so that $a_j = a_k$ if and only if the cell orbits with serial numbers j and k are in the same parcel. The "restricted growth" condition is that $a_1 = 0$ and $a_{j+1} \leq 1 + \max(a_1, \ldots, a_j)$ for $1 \leq j < n$, so that if the string is scanned from the beginning, whenever a parcel is encountered for the first time it has the lowest available number.

The assignment of parcels to sections in a distribution is encoded by a string $s_0 s_1 \ldots s_{p-1}$, where p is the number of parcels, and s_i is 0, 1, or 2 according as parcel number i belongs to D_L, D_M, or D_N.

8.3.2. *Play-by-Play of the algorithm.* The algorithm proceeds from one well-founded subset $\mathcal{S}$ to the next. The choice of x must be made so that $\mathcal{S}' = \mathcal{S} \cup \{x\}$ is well founded; in an attempt to improve performance, x is chosen so as to minimize the size of $\partial\mathcal{S}'$.

At the completion of processing for $\mathcal{S}$, these things are known:

- the number of stellations contained in $\mathcal{S}$;
- for every distribution $\mathbf{D}$ within $\partial\mathcal{S}$, the number of protostellations in $\mathrm{Nebula}(\mathbf{D})$.

The latter item is represented by a "map" whose domain is the set of $\mathbf{D}$ for which $N(\mathcal{G}, \mathbf{D}) > 0$. Let us call these the "distributions of interest."

To get from $\mathcal{S}$ to $\mathcal{S}'$, start with an empty map for stratum $\mathcal{S}'$, which would give the value 0 for $N(\mathcal{G}, \mathbf{D}')$ for every distribution $\mathbf{D}'$ within $\partial\mathcal{S}'$. Consider every distribution of interest $\mathbf{D}$ within $\partial\mathcal{S}$.

(a) Compute $\mathcal{C}_{LO}(\mathcal{S}, \mathbf{D}, x)$. If it is true, compute $\mathbf{D}' = \mathrm{Prop}_{LO}(\mathbf{D}, x)$, and increment the current value of $N(\mathcal{G}, \mathbf{D}'$ by $N(\mathcal{G}, \mathbf{D})$.

(b) Compute $\mathcal{C}_{HI}(\mathcal{S}, \mathbf{D}, x)$. If it is true, compute $\mathbf{D}' = \mathrm{Prop}_{HI}(\mathbf{D}, x)$, and increment the current value of $N(\mathcal{G}, \mathbf{D}'$ by $N(\mathcal{G}, \mathbf{D})$; also, if $\mathbf{D}' = (D'_L, D'_M, D'_N)$ and D'_L contains exactly one parcel, then add $N(\mathcal{G}, \mathbf{D})$ to the number of stellations found.

8.3.3. *First variation: count reversible and chiral stellations.* Coxeter and others[8] have set the precedent that two enantiomorphous stellations are to be counted as only one (otherwise, there would be not 59 but 86 icosahedra). The distinction can be made during the counting process that has just been outlined.

For each distribution $\mathbf{D}$. we maintain counts of the protostellations in Nebula($\mathbf{D}$) with symmetry classes "middle" and "left." In the algorithm sketched in §8.3.2, when $\mathbf{D}'$ has been found, the "left" count of $\mathbf{D}$ contributes to the left count of $\mathbf{D}'$; and the "middle" count of $\mathbf{D}$ contributes to the "left" or "middle" count of $\mathbf{D}'$ if the symmetry class of x is "left" or "middle" respectively.

8.3.4. *Second variation: exclude noncompact stellations.* To count only compact stellations, it is sufficient to restrict x to be a compact cell orbit.

8.3.5. *Third variation: no negative faces.* Some writers, e.g. Pawley[20], prefer stellations that have no negative or "re-entrant" faces. They call these **well-founded** stellations. A negative face occurs if a cell orbit x is included in the stellation, and x has an arc to a cell orbit of lower power that is not included. Given a distribution $\mathbf{D}$ within $\partial\mathcal{S}$, the "HI" step of the algorithm in §8.3.2 should be chosen only if every arc from x to a node in $\partial\mathcal{S}$ goes to a node in some parcel of D_L.

8.3.6. *Fourth variation: applying Miller's rule as interpreted.* In §6.5 mention was made of an "interpretation" of Miller's rules, by which reversible stellations are to be found with respect to the general group of symmetries and chiral stellations are to be found with respect to the special group. The cell orbit graph for the special group can be used for both tasks; but to find the reversible stellations it is necessary to add "cross-links" between enantiomorphous cell orbits. In the implementation, the presence of the cross-links matters only in computing Prop_{LO} and Prop_{HI} when the symmetry kind of x is "right."

8.4. **Some Results.** Among the polyhedra that can be expected to have large numbers of stellations, the rhombic triacontahedron has attracted the most interest. Peter W. Messer[18] seems to be the first publisher of its cell orbit graph(or "graph of cell connectivity"). He reports that John A. Gingrich and his son, Paul S. Gingrich, counted 358,833,072 stellations. Robert Webb sells computer programs, named Great Stella, at his web page `http://www.software3d.com/Stella.php`, which can (among other things) draw stellations of a great many polyhedra; he has also counted the stellations of many base polyhedra. He mentions a "misunderstanding" of the rules of §1.2 and amends the Gingriches' count to 358,833,098. Of these, 84,959 are reversible and 358,748,139 are chiral.

I have written software to count stellations, and obtained the results in Table 6. The numbers which agree with Messer and Webb are in bold face. It appears that they are following Miller's rule as interpreted in §6.5.

I have checked some of the other numbers given on Webb's site, with complete agreement. This seems to me to offer a high degree of confidence that our results are correct. I have also computed some numbers that Webb does not give: for the

	Cross-link	No Cross-link	Miller's rule, interpreted
Reversible	**84,959**	84,924	**84,959**
Chiral	438,360,690	**358,748,139**	**358,748,139**
Totals	438,445,649	358,833,063	**358,833,098**

Table 6. Stellation Counts for the Rhombic Triacontahedron

Here	*As in* [10]
0	*0*
1	*1*
2a	*2*
2b	*3*
3a	*5*
3b	*6*
3c	*4*
4a	*8*
4b	*9*
4c	*10*
4d	*7*
5a	*11*
5b	*13*
5c	*14*
5d	*12*
6a	*15*
6b	*17*
6c	*18*
6d	*16*

Here	*As in* [10]
7a	*19*
7b	*22*
7c	*23*
7d	*24*
7e	*21*
7f	*20*
8a	*25*
8b	*26*
8c	*29*
8d	*30*
8e	*28*
8f	*27*
9a	*31*
9b	*32*
9c	*33*
9d	*36*
9e	*37*
9f	*35*
9g	*34*

Here	*As in* [10]
10a	*38*
10b	*39*
10c	*40*
10d	*41*
10e	*44*
10g	*43*
10h	*42*
11a	*45*
11b	*46*
11c	*47*
11d	*48*
11g	*49*
11h	*50*
12a	*51*
12c	*52*
12f	*53*

Table 7. Facet Orbits of the Rhombic Triacontahedron

rhombicuboctahedron, there are 128,723,453,647 reversible stellations and (without cross-linking) 44,688,470,607,269,533 chiral stellations.

The computations were done on an Apple MacBook Pro, with dual 2.66GHz processors. For the rhombicuboctahedron, the computation time was about 500 seconds without cross-linking and 140 seconds with it. The number of distributions of interest seems to grow rapidly as the size of the frontiers increases. The memory requirements of a "map" with more than a million distributions are great enough to strain the virtual-memory mechanism of the computer. For this reason, when the number of distributions is large, the software writes data to a file, sorts it using the Unix *sort command, and then consolidates the results. This helps, but only to a certain extent. Indeed, when the total number of cell orbits in a constellation is greater than about 100, the counting algorithm takes hours.*

Appendix A. Labels of Facet and Cell Orbits in Earlier Works

Coxeter et al.[8] *used "clarendon type" (a bold face) for orbits which coincide with their mirror images, and for those facet orbits which always appear together with their mirror images in the cell structure. They classified the other orbits as dextro and laevo, and wrote them in Roman and italic type respectively. They did not*

Here	*As in* [18]
b	*A*
c	*B*
$\mathbf{d_2}$	*C*
$\mathbf{d_1}$	*D*
$\mathbf{e_2}$	*E*
$\mathbf{e_1}$	*F*
$\mathbf{f_3}$	*G*
$\mathbf{f_1}$	*H*
$\mathbf{f_2}$	*I*
$\mathbf{g_1}$	*J*
$\mathbf{g_2}$	*K*
$\mathbf{h_1}$	*L*
$\mathbf{h_3}$	*M*
$\mathbf{h_2}$	*N*

Here	*As in* [18]
$\mathbf{i_1}$	*O*
$\mathbf{i_3}$	*P*
$\mathbf{i_2}$	*Q*
$\mathbf{j_1}$	*R*
$\mathbf{j_2}$	*S*
$\mathbf{j_4}$	*T*
$\mathbf{j_3}$	*U*
$\mathbf{k_2}$	*V*
$\mathbf{k_4}$	*W*
$\mathbf{k_3}$	*X*
$\mathbf{l_3}$	*Y*
$\mathbf{l_2}$	*Z*
$\mathbf{l_4}$	*2A*
$\mathbf{m_3}$	*2B*

TABLE 8. Cell Orbits of the Rhombic Triacontahedron

0	a
b	0

TABLE 9. Arc Table for R30, stratum 0

1	b
c	1

TABLE 10. Arc Table for R30, stratum 1

include the non-compact facets or cells. The correspondence between their notation for facet orbits and ours is given in Table 4; Table 5 gives similar information for cell orbits.

Ede[10] *numbers the compact facet orbits of the R30, not distinguishing mirror images. The correspondence between his labels for facet orbits and ours is given in Table 7.*

Messer[18] *shows the cell orbit graph of the R30, for the general group. The correspondence between his labels for cell orbits and ours is given in table 8.*

Appendix B. Arcs in the Cell Orbit Graph of the Rhombic Triacontahedron

Tables 9 through 23 specify the arcs of the cell orbit graph of the rhombic triacontahedron. The table with a boldface number n in the upper right corner lists the cells of stratum n across the top, and those of stratum n + 1 down the side. When two cells are joined by an arc, the facet corresponding to that arc is named in the intersection of the row and column.

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$\mathit{2}$	c
d_1	2a
d_2	2b

Table 11. Arc Table for R30, stratum 2

$\mathit{3}$	d_1	d_2
e_1'	$3b'$	$3a'$
e_1''	$3b''$	$3a''$
e_2		3c

Table 12. Arc Table for R30, stratum 3

$\mathit{4}$	e_1'	e_1''	e_2
f_1	$4c''$	$4c'$	4a
f_2	$4b''$	$4b'$	
f_3			4d

Table 13. Arc Table for R30, stratum 4

$\mathit{5}$	f_1	f_2	f_3
g_1	5d		5a
g_2'	$5b'$	$5c'$	
g_2''	$5b''$	$5c''$	

Table 14. Arc Table for R30, stratum 5

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6	g_1	g_2'	g_2''
h_1	6a	6d$''$	6d$'$
h_2		6b$''$	6b$'$
h_3		6c$''$	6c$'$

TABLE 15. Arc Table for R30, stratum 6

7	h_1	h_2	h_3
i_1	7a	7f	
i_2'		7c$'$	7b$'$
i_2''		7c$''$	7b$''$
i_3'	7d$'$		7e$'$
i_3''	7d$''$		7e$''$

TABLE 16. Arc Table for R30, stratum 7

8	i_1	i_2'	i_2''	i_3'	i_3''
j_1	8a				
j_2'	8d$'$	8f$'$		8b$'$	
j_2''	8d$''$		8f$''$		8b$''$
j_3		8c$''$	8c$'$		
j_4				8e$''$	8e$'$

TABLE 17. Arc Table for R30, stratum 8

9	j_1	j_2'	j_2''	j_3	j_4
k_1	9a				
k_2'	9e$''$		9b$'$		
k_2''	9e$'$	9b$''$			
k_3'		9f$''$			9c$'$
k_3''			9f$'$		9c$''$
k_4		9d$''$	9d$'$	9g	

TABLE 18. Arc Table for R30, stratum 9

10	k_1	k_2'	k_2''	k_3'	k_3''	k_4
l_1'	10f$'$		10a$'$			
l_1''	10f$''$	10a$''$				
l_2'		10e$'$				10b$'$
l_2''			10e$''$			10b$''$
l_3		10g$'$	10g$''$	10c$''$	10c$'$	
l_4				10d$''$	10d$'$	10h

TABLE 19. Arc Table for R30, stratum 10

11	l_1'	l_1''	l_2'	l_2''	l_3	l_4
m_1	$11f''$	$11f'$			$11a$	
m_2	$11e''$	$11e'$	$11b''$	$11b'$		
m_3'			$11g'$		$11d'$	$11c'$
m_3''				$11g''$	$11d''$	$11c''$
m_4						$11h$

Table 20. Arc Table for R30, stratum 11

12	m_1	m_2	m_3'	m_3''	m_4
n_1'	$12d''$	$12e''$		$12a'$	
n_1''	$12d'$	$12e'$	$12a''$		
n_2'			$12f''$		$12b'$
n_2''				$12f'$	$12b''$
n_3			$12c''$	$12c'$	

Table 21. Arc Table for R30, stratum 12

13	n_1'	n_1''	n_2'	n_2''	n_3
o_1	$13d'$	$13d''$			$13a$
o_2	$13e'$	$13e''$	$13b''$	$13b'$	
o_3			$13c''$	$13c'$	$13f$

Table 22. Arc Table for R30, stratum 13

14	o_1	o_2	o_3
p'	$14c'$	$14b'$	$14a'$
p''	$14c''$	$14b''$	$14a''$

Table 23. Arc Table for R30, stratum 14

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